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GAUSSIAN LABELING OF PRODUCT GRAPH IS P

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ABSTRACT

This paper explores a novel graph labelling scheme using Gaussian integers applied to Cartesian product graphs. We introduce a prime labelling technique where Gaussian integers, as opposed to traditional natural numbers, are assigned to the graph elements. The primary result is a proof that this labelling is prime for the specified Cartesian product graphs. A significant application of this labelling is demonstrated in its ability to efficiently reveal Hamiltonian paths within the graph structure. Specifically, the Gaussian integer prime labelling provides a direct mechanism for identifying a Hamiltonian path that corresponds to the minimum distance. Furthermore, the chapter includes a comprehensive analysis of the graph's total weight based on this labelling. The study of such Gaussian integer labellings, including magic, anti-magic, and prime variants, is motivated by their potential for groundbreaking applications in coding theory and network design, particularly in the creation of innovative block labelling schemes. Given the widespread utility of Cartesian product graphs across diverse fields like computer science, mathematical chemistry, and biology, the proposed prime labelling with Gaussian integers is posited to offer substantial theoretical and practical advantages.

KEYWORDS: Cartesian product graph, Hamiltonian path, Gaussian integer, Minimum Distance, Prime numbers.

1. INTRODUCTION

The field of graph theory serves as a cornerstone of discrete mathematics, providing a powerful language and framework for modeling pair wise relations between objects. Its principles are foundational to computer science, chemistry, biology, and network design.[2,6,9] Within this expansive field, the study of graph labeling has emerged as a rich and dynamic area of research. A graph labeling is an assignment of integers, or more generally, elements from a set, to the vertices or edges of a graph, subject to specific conditions. The diversity of these conditions has given rise to a vast family of labeling problems, a comprehensive survey of which is meticulously maintained by.[1] highlighting the field's continual evolution and enduring appeal.

Among the most studied labeling paradigms are magic and anti-magic labeling, where the sum of labels on edges or vertices around a structure must meet a constant or injective sum requirement, respectively.[1,7] The exploration of these labeling has been extended to various graph families, including the complex structures formed by graph products.[7] Another significant and elegant branch is prime labeling, where vertices are labeled with distinct integers such that the labels assigned to adjacent vertices are relatively prime.[1,4] The inherent connection to number theory, a field deeply explored in texts like.[8] provides prime labeling with a unique mathematical depth and has spurred investigations into their existence and properties across numerous graph classes.

A particularly fertile ground for such investigations is the class of graphs formed by the Cartesian product. The Cartesian product of two graphs G and H, denoted $G\square H$, constructs a new graph whose vertex set is the Cartesian product of the vertex sets of G and H, with two vertices being adjacent if they are identical in one coordinate and adjacent in the other. This construction is a fundamental operation in graph theory, extensively detailed in seminal works like.[3] and.[9] Product graphs are indispensable in computer science for modeling parallel architectures, in coding theory for constructing codes, and in chemistry for representing molecular graphs.[3,6,10] Their structured nature often makes them ideal candidates for studying complex labeling schemes, as their regularity can be leveraged to construct systematic label assignments.[2,7]

While traditional graph labeling employs subsets of the natural numbers, recent innovative work has begun to explore labels from more algebraically rich sets. A pioneering step in this direction is the introduction of Gaussian integers as labels. Gaussian integers, complex numbers of the form a + bi where a and b are integers and $i^2 = -1$, form a unique factorization domain, a property that makes them a natural candidate for generalizing concepts like primality.[8] The work of.[4] on "Prime Gaussian Integer Labeling" of trees demonstrates the feasibility and novelty of this approach, opening a new subfield that merges algebraic number theory with graph labeling. This innovation moves beyond mere abstraction; by expanding the label set from the one-dimensional integers to the two-dimensional complex plane, we unlock a new dimension of possibilities for encoding information and analyzing graph structures.

This chapter positions itself at the confluence of these advanced ideas. We investigate a prime labeling scheme for Cartesian product graphs using Gaussian integers. Our primary objective is to prove that such a labeling is not only possible but also possesses properties that yield significant practical applications. The structured nature of Cartesian product graphs.[3,9] combined with the algebraic properties of Gaussian integers.[8] allows us to construct a labeling that is inherently prime. This investigation builds upon the foundational surveys of.[1] the structural understanding from.[2,6,9] and extends the initial forays into Gaussian integer labeling by.[4]

The utility of this novel labeling extends far beyond a theoretical existence proof. A major application we explore is in the efficient identification of Hamiltonian paths. A Hamiltonian path, which visits every vertex in a graph exactly once, is a classic and often computationally difficult problem to solve.[2,6] We demonstrate that our specific Gaussian integer prime labeling provides an implicit roadmap, from which a Hamiltonian path can be easily extracted. Furthermore, we prove that this path is not just any Hamiltonian path, but one that corresponds to the minimum distance traversed within the graph's geometric embedding implied by the labeling. This offers a direct algorithmic insight into a traditionally hard problem, leveraging the novel labeling for computational advantage.

Finally, to quantify the overall structure defined by our labeling, we calculate the total weight of the graph. This metric aggregates the labels across the graph's vertices or edges, providing a single numerical descriptor that encapsulates the lacewing’s global properties. Such calculations are crucial for applications where the graph's total "cost" or "capacity" is of interest, linking our theoretical construction back to practical metrics used in network design and optimization.[10]

In summary, this chapter makes a multifaceted contribution. It introduces and validates a new prime labelling paradigm.[12] for Cartesian product graphs using Gaussian integers.[11] It demonstrates the practical power of this paradigm by facilitating the discovery of a minimum-distance Hamiltonian path. It concludes with a comprehensive weight analysis of the labeled graph. The implications of this work are broad, potentially influencing fields that rely on efficient network design and robust coding schemes.[10] By bridging advanced graph products.[3] algebraic number theory.[8] and innovative labelling techniques.[1,4] we aim to push the boundaries of graph theory and unlock new methodologies for solving complex problems in applied mathematics and computer science.

In this paper, we are going to investigate that the labeling of Gaussian integer to the Cartesian product graphs is prime. And from this labeling it is easy find the Hamiltonian path which gives the minimum distance. Finally, the total weight of the graph is calculated.

Gaussian integer labeling creates a new type of labeling instead of a natural number. These labels are assigned to either vertices or edges to study the specific properties such as magic, anti-magic and prime labeling etc.

New Gaussian labeling variants are being studied for potential applications in coding theory, and network design creating novel block labeling schemes.

Cartesian product graphs are used in different fields such as computer science, coding theory, biology, mathematical chemistry and metric graph theory etc., and prime labeling of Gaussian integers to the product graphs give more and more applications in different fields.

1.1. Some important definitions

(1). Gaussian Integers: A Gaussian integer is a complex number a + ib where a and b both are integers. The set of Gaussian integers are denoted by Z[i] = {a + ib/a, b in Z, $i^2 = -1$}.

Where a is called a real part and b is called a imaginary part

(2) Units in Gaussian integers: whose elements in the Gaussian integer set which are having the multiplicative in that set are called units.

The units in Z[i] are 1, -1, i, -i.

(3) Relatively prime Gaussian integers: Any two Gaussian integers x and y are said to be relatively prime if the greatest common divisor (GCD) is a unit. That means the common divisors are only 1, -1, i, -i

(4) Norm of Gaussian integer: The norm of the Gaussian integer a + ib is $a^2 + b^2$. It is denoted by N(a + ib) i.e., if x = a + ib then $N(x) = a^2 + b^2$. It is always non-negative integer. Norm of a unit is always one.

(5) Gaussian vertex labeling: If the Gaussian integers are assigning to the vertices of the graph then it is called Gaussian vertex labeling.

(6) Gaussian edge labeling: If the Gaussian integers are assigning to the edges of the graph then it is called Gaussian edge labeling.

(7) Hamiltonian path: It is a simple path in a graph that visit every vertex exactly once, without retuning to the starting vertex.

(8) Path weight: The path weight is the sum of all edge weights which are included in that path.

(9) Weight of the graph: It is sum of all edge weights of the entire graph.

Both path weight and total weight of the graph are having a lot of applications in real world. The weights which are assign to the edges is may be distances, cost or time etc.

The weights are allowed for path finding the optimization problem, such as finding the shortest route on a map or determining the minimum cost or it gives the least time to travel between two points etc.,

1.2. Gaussian labelling of Cartesian Product graph $C_n \square P_2$, $n \geq 3$.

In this section we are going to find the prime labeling of the Gaussian integers to the Cartesian product of the cycle $C_n$and path $P_2$of length 2

Lemma (a): For $n \geq 3$, the Gaussian integers $(n+1)+2i, (2n-1)+2i \; and \; 2n+3i$ are relatively prime in $Z[i]$

Proof: let $Z[i]$ be the set of Gaussian integers

Given numbers $a = (n+1)+2i, \;\; b = (2n-1)+2i, \;\; c = \;2n+3i$

To prove these are relatively prime, for that we have to prove that the GCD of all these three is unity i.e., $1 \; (or) -1 \; (or) \; i \; (or) -i$

If possible, let d be the common divisor of the above Gaussian integers other than unit. We know that, if we subtract the numbers the common divisor doesn't change

So, $\quad b - 2a = (2n-1)+2i - 2[(n+1)+2i] = -3-2i$

$c - 2a = 2n+3i - 2[(n+1)+2i] = -2-i$

Now the norms $N(-3-2i) = 13$ and $N(-2-i) = 5$

Since d be the common divisor then the norm of d must divides the 13 and 5

i.e., $N(d)/_{13}$ and $N(d)/_{5} \Rightarrow N(d) = 1$

No Gaussian integer gives the norm 1 other than unity. Hence d must be unit $i.e., d = 1\ (or) - 1\ (or)\ i\ (or) - i$. So, the given numbers are relatively prime.

Lemma(b): The Gaussian integers $2n + 3i,\ 2n + 5i,\ (n+1) + 4i,\ (2n-1) + 4i$ are relatively prime in $Z[i]$.

Proof: let $Z[i]$ be the set of Gaussian integers

Let $a = 2n + 3i,\ b = 2n + 5i,\ c = (n+1) + 4i,\ d = (2n-1) + 4i$

To prove a, b, c, d are relatively prime

Consider $b - a = 2n + 5i - 2n - 3i = 2i$

Take another combination $d - 2c = 2n - 1 + 4i - 2n - 2 - 8i$

$= -3 - 4i$

We know that any GCD of a, b, c, d must divide the$2i, -3 - 4i$ that means the norm of the common divisor divide the norm of $2i, -3 - 4i$

so, $N(2i) = 4,\ N(-3-4i) = 25$ the $\gcd(4, 25) = 1$, that means only units can divide the $2i, -3 - 4i$. so, they are co-primes.

Therefore $a,\ b,\ c,\ d$ are relatively prime for all integers n. Because every common divisor of $a,\ b,\ c,\ d$must also divide the both $2i, -3 - 4i$.

Theorem (1): The Gaussian edge labeling of the Cartesian product of the cycle C_n with the path $P_2$is prime.

Proof

Let C_n be the cycle of length n and $V(C_n) = \{1, 2, 3, \ldots \ldots \ldots n\}$

Let P_2 be the path of length 2 and $V(P_2) = \{1, 2, 3\}$ both are as shown in fig. below

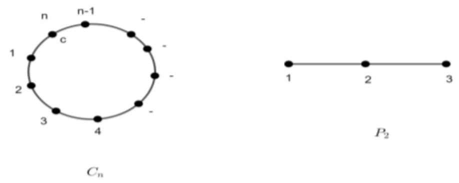
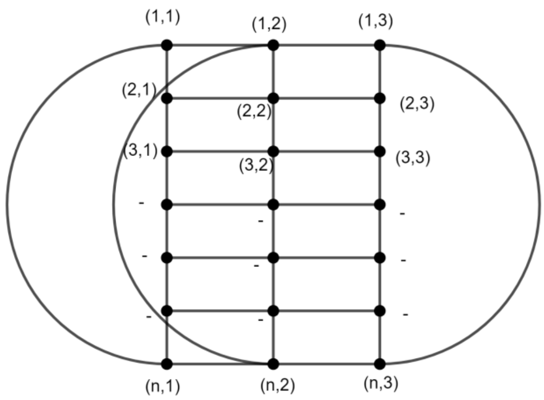


Fig.1.

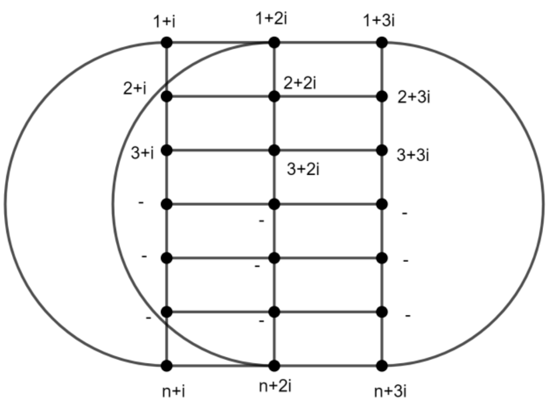
Let $G = C_n \square P_2$ be the Cartesian product graph with $3n$ vertices as shown below



$G = C_n \square P_2$

Fig.2.

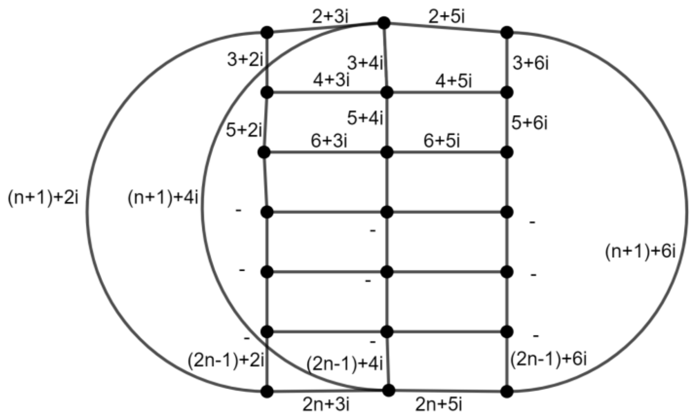
Now label the Gaussian integers for each vertex such as $(a, b) = a + ib$ then the graph G is Gaussian vertex labelling graph.



$Gaussian Vertex Labelling of G = C_n \square P_2$

Fig.3.

We know that the sum of two Gaussian integers is also Gaussian integer since $Z[i]$ satisfies closer law with respect to addition and multiplication. Using this we are going to find the edge labelling of the graph G, i.e., sum the labels of the vertices which are incidence on that edge. i.e., $w(e) = u + v$ $where$ $e = uv$ then the graph G is called Gaussian edge labelling as shown below.



$GaussianEdgeLabellingofG = C_n \square P_2$

Fig.4.

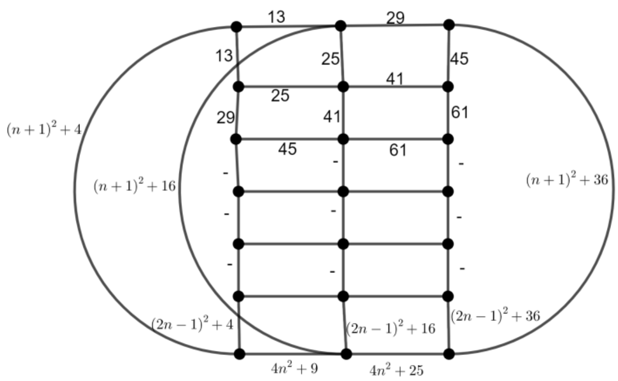
Here the graph G is prime graph, since the labels of edges incidence on each vertex are relatively prime. We say this by using lemma (a) and lemma (b).

Theorem (2): The Gaussian norms of the graph $G = C_n \square P_2$ is prime graph for $n \not\equiv 2\ or\ 9\ (mod 13)$

Proof: Let $G = C_n \square P_2$ be the Cartesian product graph with edge labelling as shown in above fig.4

The Gaussian norm of the Gaussian integer $a + ib$ is $N(a + ib) = a^2 + b^2$

After calculating the all norms of the graph in fig.4 we see that,



$NormsLabelling\ ofG = C_n \square P_2$

Fig .5.

1.3. By observing the integers incidence on some vertices is relatively prime, since the GCD is 1. And remaining vertices are having the numbers involving n,for that we have to prove that

(1) The GCD of $(13, 13, (n+1)^2+4) = 1$

Since $n \not\equiv 2 \; or \; 9 \; (mod 13)$ so $(n+1)^2+4$ is not a factor of 13 , hence their GCD is 1.

(2) The GCD of $(13, 25, 29, (n+1)^2+16) = 1$

Since $13 \; , 25 \; and \; 29$ are relatively prime so what ever may be the n the set of integers are also relatively prime. Similarly, the GCD of $(45, 29, (n+1)^2+36) = 1$.

(3) The GCD of $((n+1)^2+4, \; 4n^2+9, \; (2n-1)^2+4) = 1$

Let $a = (n+1)^2+4, \; b = 4n^2+9, \; c = (2n-1)^2+4)$

Now compute $4a - b = 4[(n+1)^2+4] - [4n^2+9]$

$= 4n^2+8n+20-4n^2-9$

$= 8n+11$

$c - b = (2n-1)^2+4-4n^2-9$

$= 4n^2-4n+5-4n^2-9$

$= -4n-4$

So, any common divisor $g = gcd(a, b, c)$ divides $8n+11 \; and \; 4n+4$. Hence g divides the integer combination of $8n+11 \; and \; 4n+4$

so, consider $8n+11-2(\; 4n+4) = 3.$

If $3 \, / \, g$ then $3/a \, , 3/b \; and \; 3 \, / \, c$. Let $3/b = 3/4n^2+9$

$\Rightarrow 4n^2+9 \equiv 0(mod\; 3)$

$\Rightarrow n^2 \equiv 0(mod\; 3)$

$\Rightarrow n \equiv 0(mod\; 3)$

But we have $a = (n+1)^2+4 = n^2+2n+5$

$\Rightarrow n^2+2n+5 = 0+0+2 \equiv 2(mod\; 3)$

$\Rightarrow 3 \nmid a$ which is a contradiction

So, $3 \nmid g$ hence $g = 1 \Longrightarrow gcd(a, b, c) = 1$

Therefore, a, b, c are relatively prime.

(4) The GCD of $((n+1)^2+16,\ 4n^2+9,\ 4n^2+25,\ (2n-1)^2+16)=1$

Let $a=(n+1)^2+16,\ b=4n^2+9,\ c=4n^2+25,\ d=(2n-1)^2+16)$

Let $g=\gcd(a,b,c,d)$ we know that any common divisor of $b\ and\ c$ divide their difference

i.e., $c-b=16$ so we get $g/16 \Rightarrow g=2\ or\ 4\ or\ 8\ or\ 16$

But for any $n, b\ and\ c$ are odd numbers so $g \nmid b\ and\ g \nmid c \Longrightarrow g=1$.

Therefore $\gcd(a,b,c,d)=1$. Hence, they are relatively prime.

(5) The GCD of $((n+1)^2+36,\ \ 4n^2+25,\ (2n-1)^2+36)=1$

Let $a=(n+1)^2+36,\ \ b=4n^2+25,\ c=(2n-1)^2+36)$

Let $g=gcd(a,b,c)$

Now $b-c=4n^2+25-4n^2+4n-37$

$=4n-12=4(n-3)$

Now $4a-b=4n^2+8n+148-4n^2-25$

$=8n+123$

Compute $(4a-b)-2(b-c)=8n+123-8n+24$

$=147$

If $g=gcd(a,b,c)$ then $g/147 \Longrightarrow g=3\ or\ 7\ or\ 21$

Now check $b=4n^2+25 \Longrightarrow n^2+1(mod\ 3) \Longrightarrow 1or\ 2\ (mod\ 3)$

$\Longrightarrow 3 \nmid b$

Check $b=4n^2+25 \Longrightarrow 4n^2+4\ (mod\ 7)$

$\Longrightarrow 4(n^2+1)(mod\ 7)$

$\Longrightarrow 7 \nmid b$

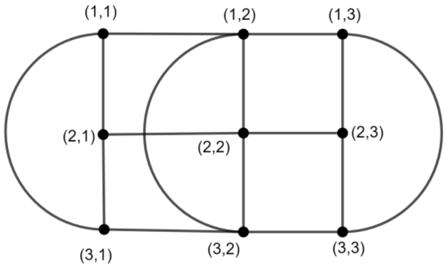
Hence the prime factors of 147 are not dividing $b\ \Longrightarrow g=1$

i.e., $gcd(a,b,c)=1$ so, they are relatively prime.

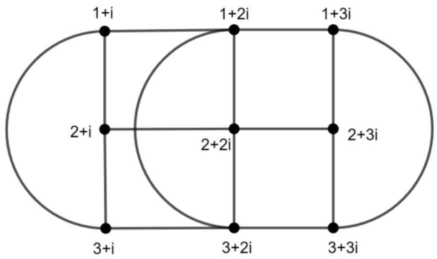
There fore, we conclude that the graph $G=C_n \square P_2$ is prime.

Examples for the labelling of Gaussian integers and their norms:

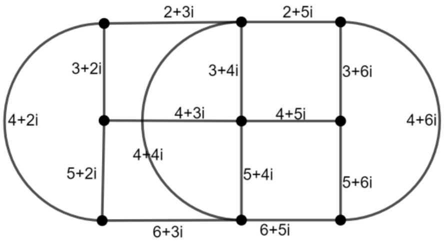
Example (1) The Cartesian product of Cycle C_3 and path P_2 i.e., $G=C_3 \square P_2$



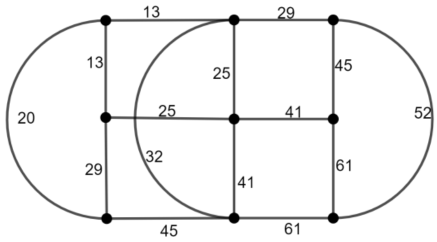
Cartesian product graph of $C_3\square P_2$



Gaussian Vertex labelling of $C_3\square P_2$



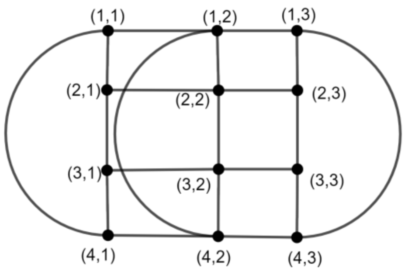
Gaussian Edge lebelling of $C_3\square P_2$



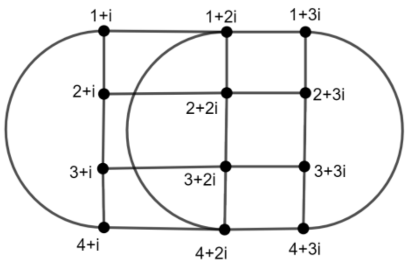
Gaussian Norms labelling of $C_3\square P_2$

In the above example(1) $G = C_3\square P_2$ the Gaussian edge labeling is prime that means the edge labels incidence on each vertex is relatively prime. Similarly, the Gaussian norms indicated on the edges are also relatively prime at each vertex.

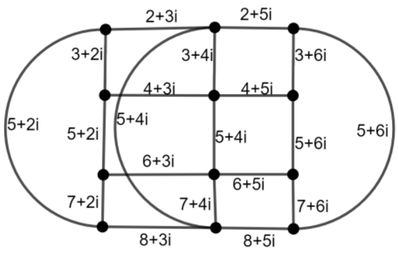
Example (2) The Cartesian product of Cycle C_4 and path $P_2$is $G = C_4\square P_2$



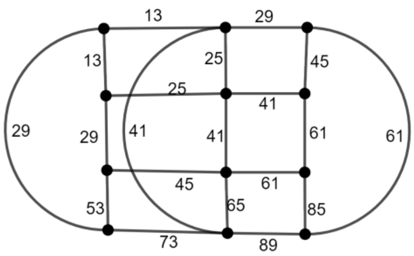
Cartesian product graph of $C_4\square P_2$



Gaussian Vertex labelling of $C_4\square P_2$



Gaussian Edge lebelling of $C_4\square P_2$



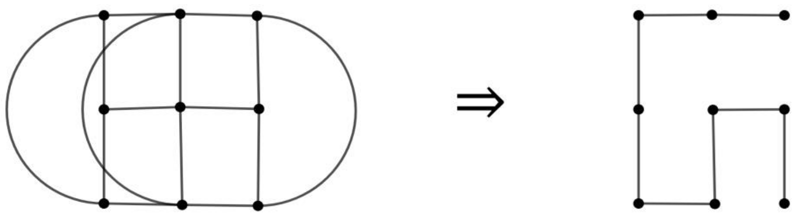
Gaussian Norms labelling of $C_4\square P_2$

In the above example(2) $G = C_4\square P_2$ the Gaussian edge labeling and the Gaussian norms indicated on the edges are relatively prime at each vertex.

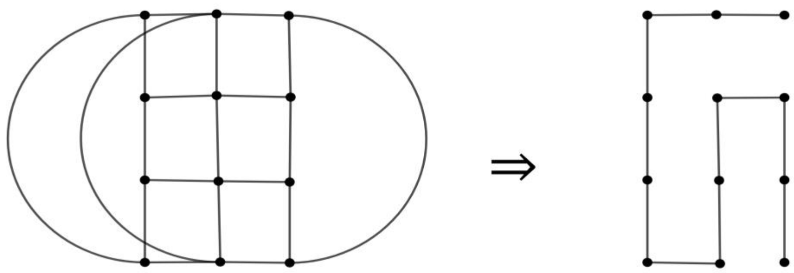
1.4: Shortest Hamiltonian path of the weighted graph $G = C_n \square P_2$

In this section the graph $G = C_n \square P_2$ as shown in fig .5 gives the norm labels as weights of the edges. Now to find the shortest Hamiltonian path of that graph i.e., which gives the minimum length of the path? Hamiltonian path means which covers all vertices of the graph exactly once.

Note: There are number of ways to cover all the vertices of the graph exactly once. But the graph $G = C_n \square P_2$ contains n rows and three columns so the total vertices are $3n$. For the shortest Hamiltonian path of $G = C_n \square P_2$ is considered in the shape of alphabet capital G. for example.



$G = C_3 \square P_2$ The shortest Hamiltonian Path of G



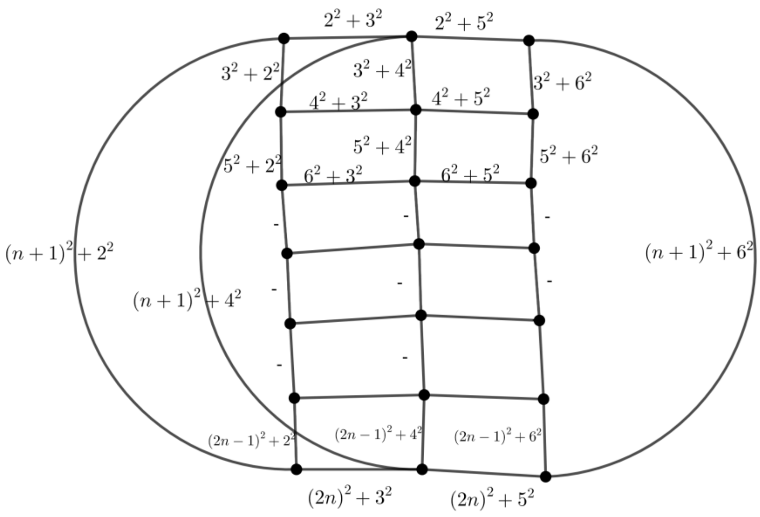
$G = C_4 \square P_2$ The shortest Hamiltonian Path of G

Theorem 3: The length of the shortest Hamiltonian path of $G = C_n \square P_2$ with Gaussian integer norms is $4n^3 + 4n^2 + 55n - 37$.[Where path is as mentioned above.]

Proof: Let $G = C_n \square P_2$ be the cartesian product of cycle C_n and path P_2 (Fig.2)

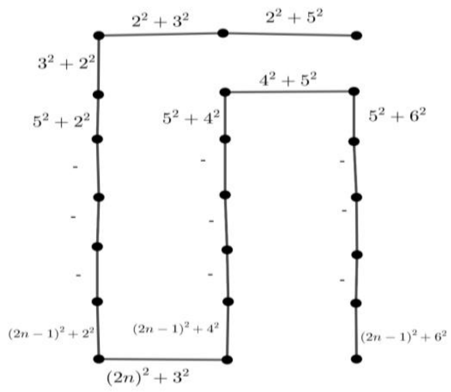
Label the Gaussian integers to the edges as shown in fig. 4

Convert the Gaussian integer into the norm and label it to the edges then the graph is given by



Gaussian norms LabellingofG $= C_n \square P_2$

Now we want to find the shortest Hamiltonian path, for this we consider a path which covers all the vertices exactly once. So, the path is given below [as mentioned in the above example].



Shortest hamiltonian path of $G = C_n \square P_2$

The length of the path

$$= \{(2^2+5^2)+(2^2+3^2)\} + \{(3^2+2^2)+(5^2+2^2)\dots\dots\dots((2n-1)^2+2^2)\} + \{(2n)^2+3^2\} + \{(5^2+4^2)+(7^2+4^2)\dots\dots\dots(2n-1)^2+4^2\} + \{4^2+5^2\} + \{(5^2+6^2)+(7^2+6^2)\dots\dots\dots(2n-1)^2+6^2\}$$

$$\Rightarrow \{(2^2+5^2)+(2^2+3^2)+(3^2+2^2)\} + \{4^2+5^2\} + 3\{5^2+7^2\dots\dots\dots\dots(n-2)terms\} + (n-2)(2^2+4^2+6^2) + \{4n^2+9$$

$$\Rightarrow \{4+25+4+9+9+4+16+25\} + 3\left\{\frac{4n^3-n-30}{3}\right\} + (n-2)56 + 4n^2+9$$

$$\Rightarrow 105 + 4n^3 - n - 30 + 56n - 112 + 4n^2$$

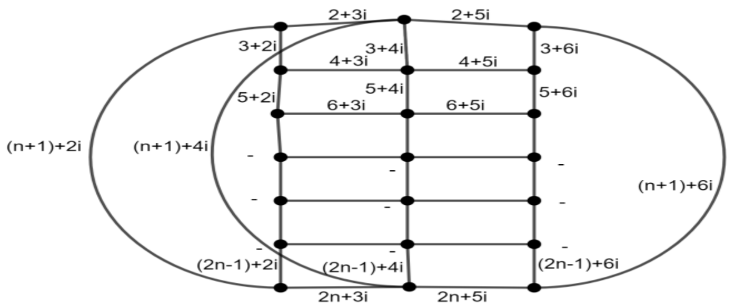
$$\Rightarrow 4n^3 + 4n^2 + 55n - 37.$$

Hence the length of the shortest Hamiltonian path is $4n^3 + 4n^2 + 55n - 37$.

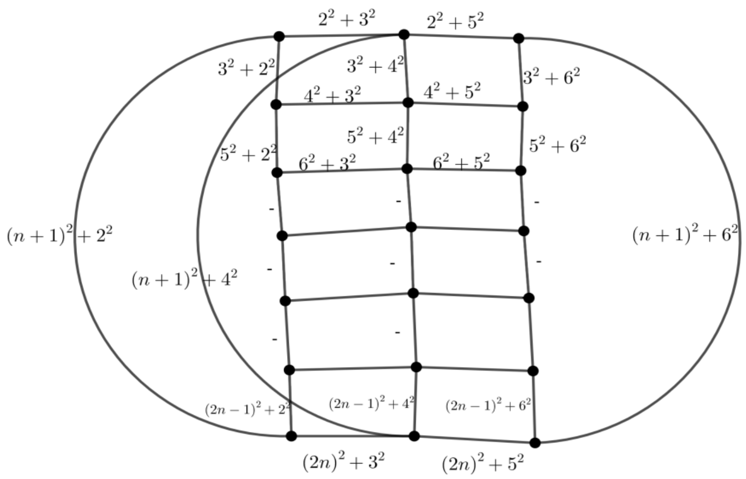
Theorem 4: The total weight of the graph $G = C_n \square P_2$ with Gaussian integer norms is $\frac{n}{3}(20n^2 + 21n + 289)$.

Proof: Let $G = C_n \square P_2$ be the Cartesian product graph of cycle and path.

The Gaussian integer labeling and norms labeling is as shown below



$Gaussian Edge Labelling of G = C_n \square P_2$



$Gaussian\ \ norms\ Labelling of G = C_n \square P_2$

Total weight of the graph is

$$= n(3^2+5^2) + n(2^2+4^2+6^2) + 3(n+1)^2 + 2(2^2+4^2+6^2 \ldots\ldots\ldots upto\ n\ terms) + 3(3^2+5^2+7^2 \ldots\ldots\ldots upto\ (n-1) terms)$$

$$\Rightarrow 34n + 56n + 3(n^2+2n+1) + 2\left[\frac{2n(n+1)(2n+1)}{3}\right] + 3\left[\frac{4n^3-n-3}{3}\right]$$

$$\Rightarrow 90n + 3n^2 + 6n + 3 + 4n^3 - n - 3 + \frac{(4n^2+4n)(2n+1)}{3}$$

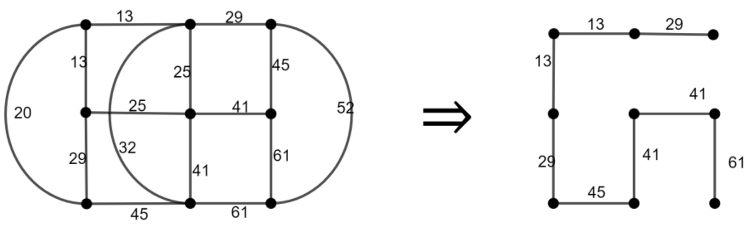
$$\Rightarrow 4n^3 + 3n^2 + 95n + \frac{(8n^3 + 12n^2 + 4n)}{3}$$

$$\Rightarrow \frac{12n^3 + 9n^2 + 285n + 8n^3 + 12n^2 + 4n}{3}$$

$$\Rightarrow \frac{n}{3}(20n^2 + 21n + 289).$$

Hence the total weight of the graph $G = C_n \square P_2$ is $\frac{n}{3}(20n^2 + 21n + 289).$

Example (3) The shortest Hamiltonian path and its length and finally the total weight of the graph $G = C_3 \square P_2$ is given by



Gaussian Norms labelling of $C_3 \square P_2$ *Shortest Path of $C_3 \square P_2$*

From above figure the total weight of the graph is 532.

By the formula total weight is $\frac{n}{3}(20n^2 + 21n + 289) = \frac{3}{3}[20(3)^2 + 21(3) + 289]$

$$= 180 + 63 + 289$$
$$= 532.$$

Now the length of the path in the figure is 272.

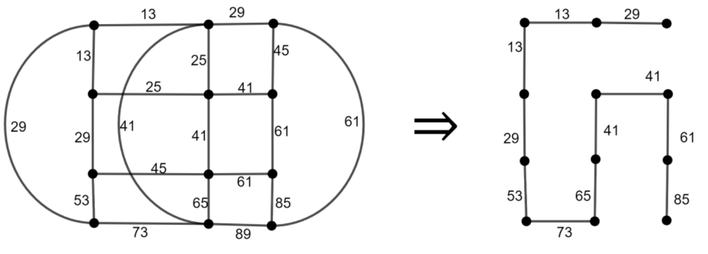
By the formula the length of the path is $4n^3 + 4n^2 + 55n - 37$

$$= 4(3)^3 + 4(3)^2 + 55(3) - 37$$

$$= 108 + 36 + 165 - 37$$

$$= 272.$$

Another example (4) the total weight and path length of the graph $G = C_4 \square P_2$



Gaussian Norms labelling of $C_4 \square P_2$ *Shortest Path of $C_4 \square P_2$*

From above figure the total weight of the graph is 924.

By the formula total weight is $\frac{n}{3}(20n^2+21n+289)=\frac{4}{3}[20(4)^2+21(4)+289]$

$$=\frac{4}{3}[320+84+289$$

$$=\frac{4}{3}(693)=924$$

Now the length of the path in the figure is 503.

By the formula the length of the path is $4n^3+4n^2+55n-37$

$$=4(4)^3+4(4)^2+55(4)-37$$

$$=256+64+220-37$$

$$=503.$$

2. CONCLUSIONS

It concludes that the Gaussian integer labeling of the Cartesian product graph and labeling with their norms is prime.

Conflicts of Interest

The Authors declared that there are no Conflicts of Interest.

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