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THE EXTENSION METHOD FOR SOLVING BOUNDARY VALUE PROBLEMS OF THE THEORY OF OSCILLATIONS OF BODIES WITH HETEROGENEITY

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ABSTRACT

The article considers the problem of determining the shifts of natural frequencies and changes of mode shape functions for low-frequency longitudinal vibration of an one-dimensional elastic rod containing local stepwise heterogeneities (defects). It is assumed that defects have small linear dimensions in comparison with the rod length and are characterized by a changes in Young's modulus. A new method to obtain the exact solution of the problem (method of solution extension), is proposed in the article. This method is like to the partial domain method and is based on the analytical extension of the solution from the uniform area to the non-uniform area by placing a point singularity on the non-uniform area. This method allows to obtain a solution of the problem for a rod with many defects with different parameters. The solution of the problem is constructed in the form of analytical series expansion according to the characteristic length of the heterogeneity which is considered small with respect to the wavelength. An infinite recursive system of boundary value problems with point discontinuities is obtained. The system allows to obtain a solution of the problem with a given accuracy.

KEYWORDS: Rod oscillation, heterogeneity, spectrum, extension method.

1. INTRODUCTION

There is an extensive literature dealing with the problem of vibrations of physically and geometrically non-uniform mechanical systems (Lia et al., 2002; Kaplunov et al., 2016; Rubio et al., 2015; Inaudi et al., 2010; Stephen et al., 2012; Weaver et al., 1990; Soloviev et al., 2018; Zrazhevsky et al., 2016). This is due to the widespread using of such systems in engineering structures. Moreover, the purpose of the study can be both the modeling of the behavior of such systems (Lia et al, 2002; Kaplunov et al, 2016; Inaudi et al., 2010; Stephen et al., 2012) and the development of equipment for diagnostic methods of the functional ability of products and structures (Rubio et al, 2015; Zrazhevsky et al., 2019). Rod systems along with shells and beams are the most common structural elements used in engineering. In some cases, the study of such rods is based on 3D models (Stephen et al., 2012). However, in many cases it is enough to consider one-dimensional approximations by applying various hypotheses of the dynamic stress-strain state.

One-dimensional approximations make sense as direct models of individual structural elements, in addition, they are used as model objects in researching of the main features of the behavior of more complex systems. In many cases, the purpose of the such objects vibration study is to determine the characteristics of the vibration spectrum, for example, to determine the natural frequencies and their dependence on the type and properties of heterogeneity. Studies affect various cases of changes in characteristics along the rod: continuous (Lia et al., 2002; Stephen et al., 2012), spasmodic (Kaplunov et al., 2016; Rubio et al., 2015) and piecewise continuous (Inaudi et al., 2010). In this case, various research methods are used - exact analytical (Lia et al., 2002; Rubio et al., 2015; Inaudi et al., 2010), numerical (Soloviev et al., 2018) and analytically approximate (Kaplunov et al., 2016; Stephen et al., 2012; Zrazhevsky et al., 2019).

This paper is devoted to the study of natural low-frequency harmonic vibrations of an elastic rod under Kirchhoff-Love hypotheses with local step wise heterogeneity (defects) in it. It is assumed that defects have small linear dimensions in comparison with the rod length and are characterized by a change in Young's modulus. The subject of the study is the frequency shifts of free vibrations and the change in vibration mode shapes. A new method for the exact solution of the problem is proposed. The method is based on the analytical extension of the solution from the uniform area to the non-uniform area by placing a point singularity in the non-uniform area. The main idea of the method is consonant with the method of integral equations (Zrazhevsky et al., 2016; Zrazhevsky et al., 2017).

2. MATERIALS AND METHODS

In a dimensionless formulation, the problem of a rod vibration with heterogeneity can be formulated as follows. Consider a rod of length 1 having a defect area D (**Figure 1**). The area D is characterized by three unknown parameters: length $2l$, location $X : (X - l; X + l)$, and change in Young's modulus $\kappa = \frac{\Delta E}{E_0} = \frac{E_1 - E_0}{E_0}$, where E_0 is the known Young modulus for the domains D_1 and D_2, E_1 is the unknown Young modulus for the area D.

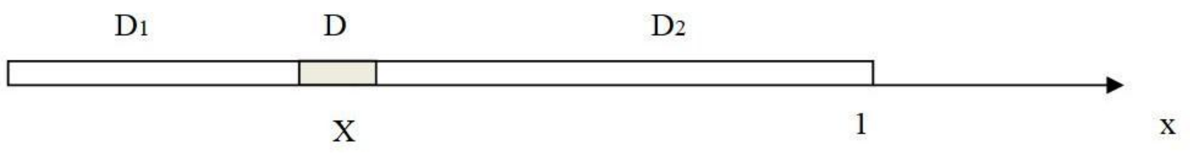

Figure 1: Geometry of the inhomogeneous rod with a defect area *D*.

Let $\varphi(x)$ is a function that describes the elastic displacements in the domains D_1 and D_2, and is a function that describes the elastic displacements in the defect area D. It is required to find $\varphi(x)$ and $\varphi_1(x)$ that satisfy the equations (Zrazhevsky et al., 2016).

$$\begin{cases} \varphi'' + k^2\pi^2\varphi = 0, \ x \in D_1 \cup D_2 \\ \varphi(0) = \varphi(1) = 0 \\ \varphi'(0) = \varphi'(1) = 0 \end{cases} \tag{1}$$

$$\varphi_1'' + \frac{k^2\pi^2}{1-\kappa}\varphi_1 = 0, \ x \in D \tag{2}$$

and stitching conditions at the boundaries of the domains D_1, D:

$$\begin{cases} \varphi(X-l) = \varphi_1(X-l) \\ \varphi(X+l) = \varphi_1(X+l) \end{cases} \tag{3}$$

and D, D_2:

$$\begin{cases} \varphi'(X-l) = (1-\kappa)\varphi_1'(X-l) \\ \varphi'(X+l) = (1-\kappa)\varphi_1'(X+l) \end{cases} \tag{4}$$

The classical approach to solve the problem (1) - (4) is to find a solution in the form of general solutions of equations (1) and (2). Satisfaction of conditions (3) and (4) leads to a homogeneous system of linear equations. The existence of a nontrivial solution of the system (the determinant of the system is 0) allows us to obtain an equation for finding the natural frequencies of rod vibration with an inhomogeneity in the form:

$$(-\kappa+2\sqrt{1-\kappa}+2)\sin\left(\pi k\left(2l\left(\frac{1}{\sqrt{1-\kappa}}-1\right)+1\right)\right)+$$

$$+(\kappa+2\sqrt{1-\kappa}-2)\sin\left(\pi k\left(1-2l\left(\frac{1}{\sqrt{1-\kappa}}+1\right)\right)\right)+$$

$$+\kappa\left(\sin\left(\pi k\left(\frac{-2l}{\sqrt{1-\kappa}}-2X+1\right)\right)-\sin\left(\pi k\left(\frac{2l}{\sqrt{1-\kappa}}-2X+1\right)\right)\right)=0 \qquad (5)$$

Solution (5) relatively k determines the natural frequencies of vibration $k_S = s+\Delta k_S, s = 1,...,$ where s is the dimensionless natural vibration frequency of a homogeneous rod, Δk_s is the shift of the s -th natural vibration frequency. The relative shifts of the first six natural frequencies for various parameters of the heterogeneity are shown in **Table 1**. As can be seen from **Table 1**, the presence of heterogeneity with a reduced stiffness leads to a negative frequency shift, with an increased one to a positive one. This fact is fully consistent with the statement of the Rayleigh theorem for natural vibration frequencies (Gantmakher, 2005) **Table 1** also shows the dependence of frequencies on the characteristics of heterogeneity, Dependence of frequency shifts on X is shown in **Figure 2**.

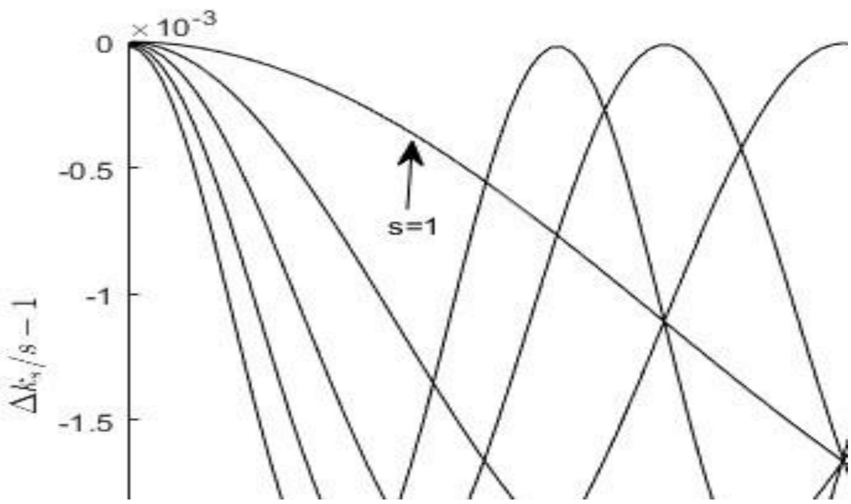

Figure 2: Dependence of frequency shift on the defect position X ($\kappa = 0.1, l = 0.01$).

As follows from the figure, the dependence of the shift of the s th natural vibration frequency on X is close to the form $\Delta k_s \approx -A_s \sin^2 s\pi X$. The dependences of the frequencies on l and κ are shown in **Figure 3** and **Figure 4** respectively.

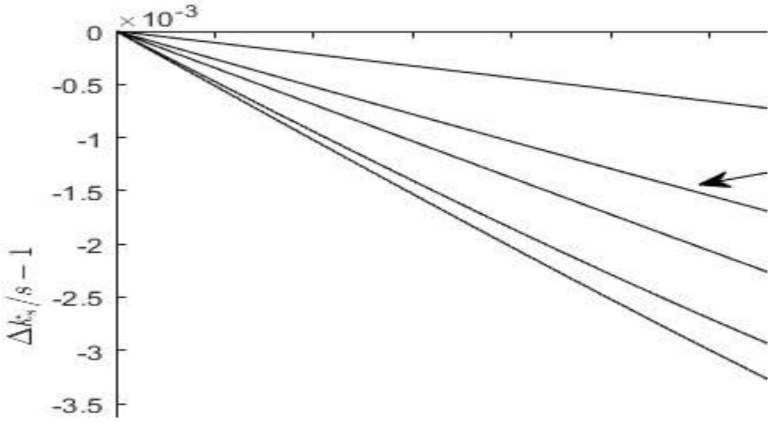

Figure 3: Dependence of frequency shift on the defect stiffness κ ($X = 0.15, l = 0.01$).

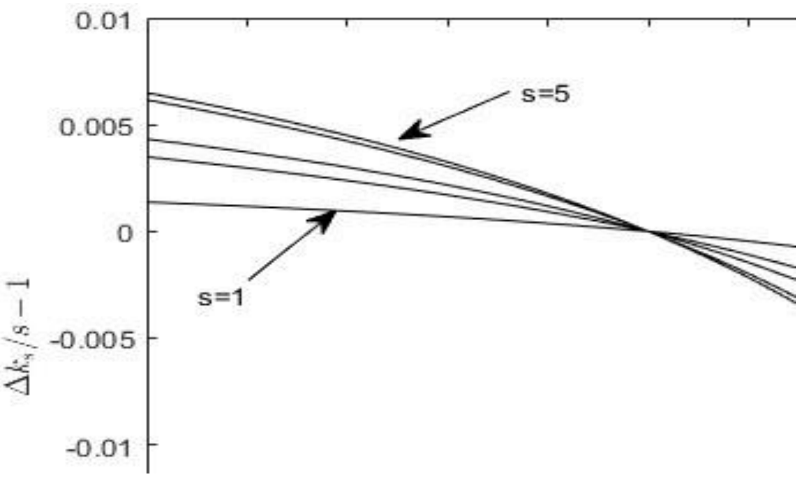

Figure 4: Dependence of frequency shift on the defect size l ($X = 0.15$, $\kappa = 0.1$).

Obviously, that the dependence of the s th natural vibration frequency shift on κ is close to the form: $\Delta k_s \approx -B_s \kappa / (1-\kappa)$, and the dependence on l is linear: $\Delta k_s \approx -C_s l$.

Table 1: Relative frequency shifts of natural vibrations for various parameters of heterogeneity.

(X ,l,κ) \ s	1	2	3	4	5	6
(0.11,0.003,0.1)	-0.00078	-0.00275	-0.00490	-0.00611	-0.00586	-0.00450
(0.21,0.003,0.1)	-0.00252	-0.00617	-0.00537	-0.00166	-0.00062	-0.00352
(0.11,0.003,0.2)	-0.00177	-0.00627	-0.01114	-0.01360	-0.01262	-0.00943
(0.11,0.003,-0.1)	0.00064	0.00221	0.00394	0.00504	0.00508	0.00409
(0.11,0.003,-0.2)	0.00116	0.00402	0.00717	0.00926	0.00949	0.00779

Table 2: Relative errors of relative frequency shifts of natural frequencies for 8 successive approximations with $X = 0.15$, $\kappa = 0.1$, $l = 0.03$.

	1	2	3	4	5	6	7
1	0.0188	0.0024	-0.0240	-0.0570	-0.0865	-0.0857	0.0321
2	-2.0*10^{-5}	6.0*10^{-5}	0.0007	0.0024	0.0058	0.0099	0.0062
3	-2.0*10^{-5}	5.0*10^{-5}	0.0006	0.0023	0.0054	0.0091	0.0047
4	-3.61*10^{-8}	-1.35*10^{-6}	-1.0*10^{-5}	-0.0001	-0.0003	-0.0008	-0.0021
5	-3.29*10^{-8}	-1.14*10^{-6}	-1.0*10^{-5}	-0.0001	-0.0002	-7.0*10^{-4}	-0.0017
6	1.3810^{-10}	1.27*10^{-8}	2.22*10^{-7}	1.93*10^{-6}	1.0*10^{-5}	0.0001	0.0002
7	1.29*10^{-10}	1.05*10^{-8}	1.67*10^{-7}	1.38*10^{-6}	8.27*10^{-6}	4.0*10^{-5}	0.0002

3. RESULTS

3.1. The extension method. A single defect

Show that finding a function in the defect area D can be reduced to finding $\varphi_1(x)$ and $\varphi_1'(x)$ at the point X. Since the function $\varphi_1(x)$ is analytic in D, $\varphi_1(x)$ is searched in the form of an analytical series:

$$\varphi_1(x) = \sum_{k=0}^{\infty} \frac{\varphi_1^{(k)}(X)}{k!}(x-X)^k .$$

Given that $\varphi_1^{(2s)}(X) = (-1)^s \left(\frac{k^2\pi^2}{1-\kappa}\right)^s \varphi_1(X)$, $\varphi_1^{(2s+1)}(X) = (-1)^s \left(\frac{k^2\pi^2}{1-\kappa}\right)^s \varphi_1'(X)$,

representation of $\varphi_1(x)$ can be written as:

$$\varphi_1(x) = \varphi_1(X)\cos\frac{k\pi(x-X)}{\sqrt{1-\kappa}} + \frac{\sqrt{1-\kappa}}{k\pi}\varphi_1'(X)\sin\frac{k\pi(x-X)}{\sqrt{1-\kappa}}. \tag{6}$$

After differentiating the system (6) and setting $x = X \pm l$ obtain:

$$\begin{cases} \varphi_1(X \pm l) = \varphi_1(X)\cos\dfrac{k\pi l}{\sqrt{1-\kappa}} \pm \dfrac{\sqrt{1-\kappa}}{k\pi}\varphi_1'(X)\sin\dfrac{k\pi l}{\sqrt{1-\kappa}} \\ \varphi_1'(X \pm l) = \mp\dfrac{k\pi}{\sqrt{1-\kappa}}\varphi_1(X)\sin\dfrac{k\pi l}{\sqrt{1-\kappa}} + \varphi_1'(X)\cos\dfrac{k\pi l}{\sqrt{1-\kappa}} \end{cases}. \tag{7}$$

After introduction of the jump function $[f]_l = \frac{f(X+l)-f(X-l)}{2}$, the average function $\mu_l f = \frac{f(X+l)+f(X-l)}{2}$ and considering that $[\varphi]_l = [\varphi_1]_l$, $[\varphi']_l = (1-\kappa)[\varphi_1']_l$, $\mu_l\,\varphi = \mu_l\,\varphi_1$, $\mu_l\,\varphi' = (1-\kappa)\mu_l\,\varphi_1'$, conditions (7) can be written as a system of difference equations for the jump and average functions in the form:

$$\begin{cases} [\varphi]_l = \dfrac{l}{k\pi\sqrt{1-\kappa}}\, tg\,\dfrac{k\pi l}{1-\kappa}\,\mu_l\,\varphi' \\ [\varphi']_l = -k\pi\sqrt{1-\kappa}\, tg\,\dfrac{k\pi l}{\sqrt{1-\kappa}}\,\mu_l\,\varphi \end{cases}. \tag{8}$$

Extending the solution from D_1 and D_2 to D allows us to reduce the problem to the following: it is necessary to find a function $\varphi(x)$ that for all $x \in (0,1) \setminus \{X\}$ satisfies equation (1) and conditions (8). Note that in this case the continuity of the first and higher order derivatives for $\varphi(x)$ at the point X is not required.

We look for $\varphi(x)$ in the form of the analytical series expansion. After finding the expressions for the expansion coefficients, the representation for $\varphi(x)$ is given by:

$$\varphi(x) = \varphi(X)\cos k\pi(x - X) + \frac{1}{k\pi}\varphi'(X)\sin \pi(x - X).$$

Then the difference equations (8) can be written as:

$$\begin{cases} [\varphi]_l = [\varphi]\cos k\pi l + \dfrac{1}{k\pi}\mu\varphi'\sin k\pi l \\ [\varphi']_l = -k\pi\,\mu\varphi\sin k\pi l + [\varphi']\cos k\pi l \end{cases}, \qquad (9)$$

$$\begin{cases} \mu_l\,\varphi = \mu\varphi\,\cos k\pi l + \dfrac{1}{k\pi}[\varphi']\sin k\pi l \\ \mu_l\,\varphi' = -k\pi[\varphi]\sin k\pi l + \mu\varphi'\cos k\pi l \end{cases}, \qquad (10)$$

where [] and μ denote jump function and average function at the point X respectively. Therefore, the problem was reduced to finding a function $\varphi(x, l)$ that satisfies equation (1) for $x \in (0,1) \setminus \{X\}$, and conditions (9) (10) at the point X. Note that (9) and (10) are valid for any l and κ.

We are looking for a function $\varphi(x,l)$ in the form of an expansion in an analytic series in powers of l in the space of generalized functions:

$$\varphi(x,l) = \sum_{s=0}^{\infty} \Phi_s(x)\, l^s \ . \qquad (11)$$

The coefficients $\Phi_s(x)$ are independent of l. To find them, it is necessary to substitute $\varphi(x,l)$ in the form (11) into the equation (1) and conditions (9), (10) and compose a system, equating the coefficients at the same powers of l. Thus $\Phi_s(x)$ can be from a system of recurrence equations:

$$\begin{cases}[\Phi_0]=0\\ [\Phi_0']=0\end{cases},\quad \begin{cases}[\Phi_1]=\dfrac{\kappa}{1-\kappa}\mu\Phi_0'\\ [\Phi_1']=0\end{cases},\quad \begin{cases}[\Phi_2]=\dfrac{\kappa}{1-\kappa}\mu\Phi_1'\\ [\Phi_2']=0\end{cases},\tag{12}$$

$$\begin{cases}[\Phi_3]=-\dfrac{\kappa(1+\kappa)}{3(1-\kappa)^2}k^2\pi^2\mu\Phi_0'+\dfrac{\kappa}{1-\kappa}\mu\Phi_2'\\ [\Phi_3']=-\dfrac{k^4\pi^4\mu\Phi_0}{3(1-\kappa)}\end{cases} \quad \text{etc.}$$

In total, eight members of series are obtained in the work.

As an example, let us find the first three approximations. We are looking for expansion coefficients $\Phi_0(x)$, $\Phi_1(x)$, $\Phi_2(x)$. Since Φ_0 and Φ_0' have zero jumps, $\Phi_0(x)$ can be found as a solution to the equation $\dfrac{d^2\Phi_0}{dx^2}+k^2\pi^2\Phi_0=0$, namely: $\Phi_0=A_0\cos k\pi x+B_0\sin k\pi x,\ \ x\in(0,1)$.

To find the first approximation, we seek a solution to the equation $\dfrac{d^2\Phi_1}{dx^2}+k^2\pi^2\Phi_1=0, x\neq X$ that satisfies the initial conditions: $[\Phi_1]=\dfrac{2\kappa}{1-\kappa}\mu\Phi_0',[\Phi_1']=0$. Consider the two terms in the expansion (11): $\varphi\approx\Phi_0+l\Phi_1$. Then $\varphi=A_0\cos k\pi x+B_0\sin k\pi x+\dfrac{2\kappa l}{1-\kappa}\mu\Phi_0'\Phi_1^*$, where

$$\Phi_1^*=\begin{cases}0, & x<X\\ \cos k\pi X\cos k\pi x+\sin k\pi X\sin k\pi x, & x>X\end{cases}.$$

Satisfying the boundary conditions $\varphi'(0)=\varphi'(1)=0$, obtains: $B_0=0$, A_0 is arbitrary. Therefore, the frequency equation in the first approximation is:

$$-\sin k\pi+2\frac{\kappa l}{1-\kappa}k\pi\ \sin k\pi X\sin k\pi(1-X)=0.\tag{13}$$

The expression for the vibration mode in the two-term approximation can be expressed as:

$$\varphi(x)=\begin{cases}0, & x<X\\ A_0\cos k\pi x+A_0\dfrac{\kappa l}{1-\kappa}(-k\pi)\sin k\pi x\ \cos k\pi(x-X), & x>X\end{cases}\tag{14}$$

and has an error $\sim l^3$ with respect to the exact value. Arguing in a similar way, a three-term approximation of the frequency equation can be obtained in the form:

$$\sin k\pi - \frac{2k\kappa l}{1-\kappa}\sin k\pi X \sin k\pi(1-X) - \frac{2k^3\kappa l^3\pi^3}{3(1-\kappa)^2}(\kappa\cos k\pi - \cos k\pi(1-2X)) = 0 \quad (15)$$

The relative errors of the solutions of the equations (13), (15) as well as the following five approximations, compared to the exact values (5) are given in **Table 2**.

It is easy to see that the accuracy decreases with an increase of a natural frequency number. Moreover, for the first five frequencies, the one-term approximation provides accuracy within 0.1, the three-term approximation provides within 0.006, and the five-term approximation provides within 0.0003. Therefore, for practical purposes, the three-term approximation (15) is optimal. The vibration modes obtained by the proposed extension method (second approximation) in comparison with the exact values are shown in **Figure 5**.

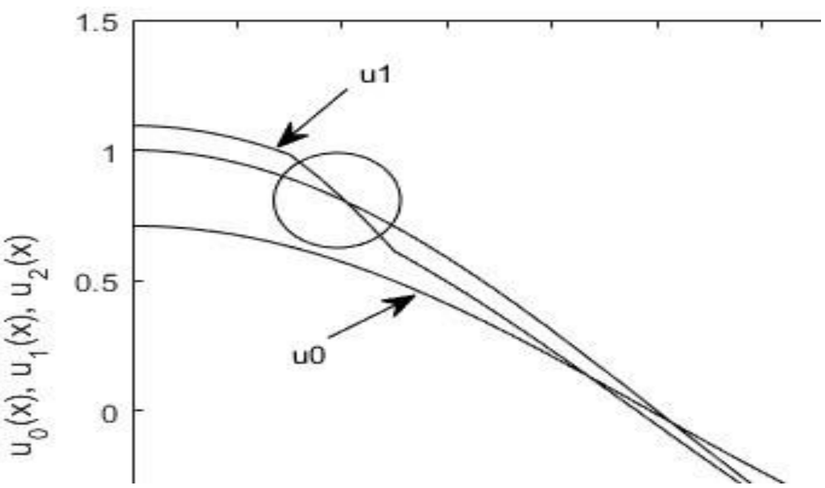

Figure 5: Displacements for the first vibration mode.

The function $u_0(x)$ describes the vibration mode of a uniform rod, $u_1(x)$ and $u_2(x)$ describe the vibration mode of a non-uniform rod with characteristics of the area $X = 0.2, \kappa = 0.5, l = 0.05$ and $X = 0.8, \kappa = -0.5, l = 0.05$ respectively.

3.2. The extension method Multiple defects

The proposed method can be easily applied for solving the problem (1) - (4) when an arbitrary finite number of defects I with parameters $\{X_i, \kappa_i, l_i\}_{i=1}^{I}$ are located in the rod with the assumption: $\max_{i=1,I}(l_i) << 1$. In this case, the recursive sequence of boundary value *i*=1, *I* problems (12) takes the form:

$$\begin{cases}[\Phi_0]|_{X_i}=0\\ [\Phi_0']|_{X_i}=0\end{cases}, \quad \begin{cases}[\Phi_1]|_{X_i}=\dfrac{\kappa_i}{1-\kappa_i}\mu\Phi_0'|_{X_i}\\ [\Phi_1']|_{X_i}=0\end{cases}, \quad \begin{cases}[\Phi_2]|_{X_i}=\dfrac{\kappa_i}{1-\kappa_i}\mu\Phi_1'|_{X_i}\\ [\Phi_2']|_{X_i}=0\end{cases}, \qquad (16)$$

$$\begin{cases}[\Phi_3]|_{X_i}=-\dfrac{\kappa_i(1+\kappa_i)}{3(1-\kappa_i)^2}\kappa_i^2\pi^2\mu\Phi_0'|_{X_i}+\dfrac{\kappa_i}{1-\kappa_i}\mu\Phi_2'|_{X_i}\\ [\Phi_3']|_{X_i}=-\dfrac{\kappa_i^4\pi^4\mu\Phi_0|_{X_i}}{3(1-\kappa_i)}\end{cases} \quad i=1,\ldots,I \quad \text{etc.}$$

The expressions for determining the natural frequencies for the first approximation (equation (13)) and the third approximation (equation (15)) are found as:

$$-\sin k\pi+2k\pi\sum_{i=1}^{I}\frac{\kappa_i l_i}{1-\kappa_i}\sin k\pi X_i\sin k\pi(1-X_i)=0 \qquad (17)$$

and

$$\begin{aligned}&\sin k\pi-2k\pi\sum_{i=1}^{I}\frac{\kappa_i l_i}{1-\kappa_i}\sin k\pi X_i\sin k\pi(1-X_i)-\\ &-\ 2/3k^3\pi^3\sum_{i=1}^{I}\frac{\kappa_i l_i^3}{(1-\kappa_i)^2}(\kappa_i\cos k\pi-\cos k\pi(1-2X_i))=0\end{aligned}, \qquad (18)$$

respectively.

4. CONCLUSIONS

A new method for solving boundary-value problems in the theory of oscillations of bodies with inhomogeneities is proposed — the extension method. The extension method is based on expanding the solution in a series in powers of parameter of the heterogeneity. The method is considered as an example of finding the dependence of the natural frequency shifts of a linearly elastic rod with heterogeneity on the characteristics of heterogeneity. The essence of the method is to extend the solution of the Helmholtz equation to the non-uniform area. The expressions of the dependence of frequency shifts on the characteristics of heterogeneity obtained using the extension method are significantly simpler in comparison with the expressions obtained by the classical approach applied, for example, in Kaplunov et al. (2016), Rubio et al., (2015).

The results of the article can be used in engineering practice to determine the spectra of bar structural elements with local inhomogeneities, as well as to obtain calculation formulas for determining the characteristics of defects in spectral methods of non-destructive testing. The proposed in the article extension method can be applied to solve problems with an arbitrary model of heterogeneity and any number of heterogeneities. The method can be directly used to solve oscillation problems of other one-dimensional models and can also be generalized to the case of two and three-dimensional problems.

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