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# SOME TENSORS IN GENERALIZED $\mathcal{B}\,R-$ RECURRENT FINSLER SPACE

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**ABSTRACT**

The generalized $\mathcal{B}\,R-$ recurrent Finsler space has been introduced by Qasem and Abdallah.[5] Now, in this paper, two theorems related to the above mentioned space have been established and proved.

**KEYWORDS:** Generalized $\mathcal{B}\,R-$ recurrent Finsler space, Berwald's covariant derivative.

## INTRODUCTION AND PRELIMINARIES

The recurrence property and generalized recurrence property have been studied by the Riemannian and Finslerian geometrics. Ruse.[10] considered the three dimensional Riemannian space having the recurrent of curvature tensor, he called such space as Riemannian space of recurrent curvature. This space has extended to $n$ –dimensional Riemannian space by Walker, Wong, Wong and Yano and others.[4,13,14] This idea was extended to Finsler space by Moor.[5] for the first time.

Pandey et al.[12] Qasem and Abdallah.[6] Qasem and Baleedi.[7] and Alaa et al.[2,3] introduced the generalized recurrent Finsler spaces for $H^i_{jkh}$, $R^i_{jkh}$, $K^i_{jkh}$ and $P^i_{jkh}$, respectively. Also, the generalized property for normal projective curvature tensor $N^I$ in sense of Berwald has been introduced by.[8]

Let $F_n$ be an $n$ –dimensional Finsler space equipped with the metric function $F(x, y)$

satisfying the request conditions.[9] The vector $y_i$ is defined by.

(1.1) $y_i = g_{ij}(x,y)y^j.$

Two sets of quantities $g_{ij}$ and its associative $g^{ij}$, which are connected by

(1.2) $g_{ij}g^{ik} = \delta_j^k = \begin{cases} 1 \ if \ j = k, \\ 0 \ if \ j \neq k. \end{cases}$

In view of (1.1) and (1.2), we have

(1.3) a) $\delta_k^i y_i = y_k$, b) $\delta_k^i y^k = y^i$ and c) $\delta_j^i g_{ir} = g_{jr}$.

The tensor $C_{ijk}$ that is known as $(h)hv-torsion\ tensor$ defined as [11]

$$C_{ijk} = \frac{1}{2}\dot{\partial}_i\, g_{jk} = \frac{1}{4}\dot{\partial}_i\dot{\partial}\,\dot{\partial}_k\, F^2$$

It is positively homogeneous of degree $-1$ in $y^i$ and symmetric in all its indices. The above tensor $C_{ijk}$ satisfies

(1.4) a) $C_{ijk}\, y^i = C_{kij}\, y^i = C_{jki}\, y^i = 0$ and b) $C_{ijk}\delta_h^k = C_{ijh}$.

Berwald's covariant derivative $\mathcal{B}_k T_j^i$ of an arbitrary tensor field $T_j^i$ with respect to $x^k$ is given by [1, 9]

$$\mathcal{B}_k T_j^i = \partial_k T_j^i - (\dot{\partial}_r T_j^i) G_k^r + T_j^r G_{rk}^i - T_r^i G_{jk}^r.$$

Berwald's covariant derivative $\mathcal{B}_k T_j^i$ appears as $T_{j(k)}^i$. Berwald's covariant derivative of the vector $y^i$ and metric tensor $g_{ij}$ satisfy

(1.5) a) $\mathcal{B}_k y^i = 0$ and b) $\mathcal{B}_k\, g_{ij} = -2C_{ijk|h}y^h = -2y^h\mathcal{B}_h C_{ijk}$.

The $h-$ curvature tensor (Cartan's third curvature tensor) is defined by

$$R_{jkh}^i = \partial_h \Gamma_{jk}^{*i} + (\partial_l \Gamma_{jk}^{*i}) G_h^l + C_{jm}^i (\partial_k G_h^m - G_{kl}^m G_h^l) + \Gamma_{mk}^{*i} \Gamma_{jh}^{*m} - k/h^* .$$

This tensor satisfies the following relations

(1.6) $R_{jki}^i = R_{jk}$.

The curvature tensor $R_{jkh}^i$, its associative $R_{rjkh}$, $R$-Ricci tensor $R_{jk}$, curvature vector $R_k$ and $h(v)-$ torsion tensor $H_{kh}^i$ satisfy

(1.7) $R_{rjkh} = R^i_{jkh} g_{ri}$

(1.8) $R_{jk} y^j = R_k$

(1.9) $R^i_{jkh} y^j = H^i_{kh} = K^i_{jkh} y^j.$

The $h(v) -$ torsion tensor satisfies the relation

(1.10) $H^i_{kh} y^k = H^i_h = -H^i_{hk} y^k,$

where $h(v)$ −torsion tensor $H^i_{kh}$ and deviation tensor $H^i_h$ are positively homogenous of degree one and two in $y^i$, respectively. The curvature vector $H_k$ and curvature scalar $H$ satisfy the following

(1.11) a) $H^i_{ji} = H_j$ and b) $H = \frac{1}{n-1} H^r_r.$

The curvature tensor $R^i_{jkh}$ and its associative tensor $R_{ijkh}$ satisfy the following identities which known as *Bianchi identity* [9]

(1.12) a) $R^r_{ijk|h} + R^r_{ihj|k} + R^r_{ikh|j} + \left(R^s_{mkh} P^r_{ijs} + R^s_{mjk} P^r_{ihs} + R^s_{mhj} P^r_{iks}\right) y^m = 0$

b) $R_{ijkh} + R_{ihkj} + R_{ikjh} + C_{ijs} H^s_{hk} + C_{ihs} H^s_{kj} + C_{iks} H^s_{jh} = 0,$

where $P^i_{jkh}$ is called *hv −curvature tensor* (*Cartan's second curvature tensor*) is defined by [8]

$$P^i_{jkh} = \dot{\partial}_h \Gamma^{*i}_{jk} + C^i_{jr} P^r_{kh} - C^i_{jh|k} ,$$

which satisfies the relations

(1.13) $P^i_{jkh}\, y^j = \Gamma^{*i}_{jkh} y^j = P^i_{kh} = C^i_{kh|r} y^r,$

where $P^i_{kh}$ called $v(hv)$ −torsion tensor.

A Finsler space $F_n$ which Cartan's third curvature tensor $R^i_{jkh}$ satisfies the condition [6]

(1.14) $\mathcal{B}_m R^i_{jkh} = \lambda_m R^i_{jkh} + \mu_m\left(\delta^i_j g_{kh} - \delta^i_k g_{jh}\right) , \quad R^i_{jkh} \neq 0,$

called a *generalized $\mathcal{B}R$ − recurrent Finsler space* and denoted it briefly by $G(\mathcal{B}R) - RF_n$.

Transvecting the condition (1.14) by $g_{il}$, using (1.5b), (1.7) and (1.3c),we get

(1.15) $\mathcal{B}_m R_{jlkh} = \lambda_m R_{jlkh} + \mu_m\left(g_{jl} g_{kh} - g_{kl} g_{jh}\right) + 2R^i_{jkh} y^h \mathcal{B}_h C_{ilm}.$

Contracting the indices $i$ and $h$ in the condition (1.14), using (1.6) and (1.3c), we get

(1.16) $\mathcal{B}_m R_{jk} = \lambda_m R_{jk}.$

Transvecting (1.16) by $y^j$, using (1.5a) and (1.8), we get

(1.17) $\mathcal{B}_m R_k = \lambda_m R_k .$

## 2. Main Results

In this section, we discuss two theorems related to generalized $\mathcal{B}R$ – recurrent space. Let us consider a $G(\mathcal{B}R) - RF_n$ which characterized by the condition (1.14).

Transvecting the condition (1.14) by $y^j$, using (1.5a), (1.9), (1.3b) and (1.1), we get

(2.1) $\mathcal{B}_m H^i_{kh} = \lambda_m H^i_{kh} + \mu_m (y^i g_{kh} - \delta^i_k y_h)$.

Further, transvecting (2.1) by $y^k$, using (1.5a), (1.10), (1.1) and (1.3b), we get

(2.2) $\mathcal{B}_m H^i_h = \lambda_m H^i_h$.

Contracting the indices $i$ and $h$ in (2.1), using (1.11a), (1.1) and (1.3a), we get

(2.3) $\mathcal{B}_m H_k = \lambda_m H_k$.

Contracting the indices $i$ and $h$ in (2.2), using (1.11b), we get

(2.4) $\mathcal{B}_m H = \lambda_m H$.

From (2.2), (2.3) and (2.4), we conclude

**Theorem 2.1.** *In $G(\mathcal{B}R) - RF_n$, the deviation tensor $H^i_h$, curvature vector $H_k$ and curvature scalar $H$ behave as recurrent.*

We know that the associate curvature tensor $R_{ijkh}$ of three dimensional Finsler space is given by the form [9]

(2.5) $R_{ijkh} = g_{ik} L_{jh} + g_{jh} L_{ik} - k/h$,

where

(2.6) $L_{ik} = \frac{1}{n-2}(R_{ik} - \frac{r}{2} g_{ik})$

and

$$r = \frac{1}{n-1} R^i_i .$$

Differentiating (2.6) covariantly with respect to $x^m$ in sense of Berwald, using (1.16) and (1.5b), we get

(2.7) $\mathcal{B}_m L_{ik} = \frac{1}{n-2}(\lambda_m R_{ik} + y^h \mathcal{B}_h C_{ikm})$.

Taking $\mathcal{B}$ – covariant derivative for eq. (2.5) with respect to $x^m$ and using eq. (1.15), we get

$$\mathcal{B}_m(g_{ik}L_{jh} + g_{jh}L_{ik} - k/h) = \lambda_m R_{jlkh} + \mu_m(g_{jl}g_{kh} - g_{kl}g_{jh})$$
$$+2R^i_{jkh}y^h\mathcal{B}_h C_{ilm} ,$$

Using eq. (2.5) in above equation, we get

(2.8) $$\mathcal{B}_m(g_{ik}L_{jh} + g_{jh}L_{ik} - k/h) = \lambda_m(g_{ik}L_{jh} + g_{jh}L_{ik} - k/h)$$
$$+\mu_m(g_{jl}g_{kh} - g_{kl}g_{jh}) + 2R^i_{jkh}y^h\mathcal{B}_h C_{ilm}.$$

Thus, we conclude

**Theorem 2.2.** *In $G(\mathcal{B}R) - RF_n$, Berwald's covariant derivative of first order for the tensors $L_{ik}$ and $(g_{ik}L_{jh} + g_{jh}L_{ik} - k/h)$ are given by eqs. (2.7) and (2.8), respectively.*

Differentiating (1.12b) covariantly with respect to $x^m$ in sense of Berwald, we get

$$\mathcal{B}_m R_{ijkh} + \mathcal{B}_m R_{ihkj} + \mathcal{B}_m R_{ikjh} + (\mathcal{B}_m C_{ijr})H^r_{hk} + C_{ijr}(\mathcal{B}_m H^r_{hk})$$
$$+(\mathcal{B}_m C_{ihr})H^r_{kj} + C_{ihr}(\mathcal{B}_m H^r_{kj}) + (\mathcal{B}_m C_{ikr})H^r_{jh} + C_{ikr}(\mathcal{B}_m H^r_{jh}) = 0.$$

Using (1.15) and (2.1) in above equation, we get

$$\lambda_m(R_{ijkh} + R_{ihkj} + R_{ikjh} + C_{ijr}H^r_{hk} + C_{ihr}H^r_{kj} + C_{ikr}H^r_{jh})$$
$$+\mu_m(\, g_{ik}g_{jh} - \ g_{jk}g_{ih}) + (\mathcal{B}_m C_{ijr})H^r_{hk} + (\mathcal{B}_m C_{ihr})H^r_{kj} + (\mathcal{B}_m C_{ikr})H^r_{jh}$$
$$+\mu_m(C_{ijr}y^r g_{hk} - C_{ijr}\delta^r_h y_k + C_{ihr}y^r g_{kj} - C_{ihr}\delta^r_k y_j + C_{ikr}y^r g_{jh} - C_{ikr}\delta^r_j y_h) = 0.$$

Using (1.12b) and (1.4) in above equation, we get

(2.10) $$(\mathcal{B}_m C_{ijr})H^r_{hk} + (\mathcal{B}_m C_{ihr})H^r_{kj} + (\mathcal{B}_m C_{ikr})H^r_{jh} - \mu_m(C_{ijh}y_k + C_{ihk}y_j$$
$$+C_{ikj}y_h + g_{jk}g_{ih} - g_{ik}g_{jh}) \ = 0\,.$$

From (1.12a), the Bianchi identity for Cartan's third curvature tensor $R^i_{jkh}$ in since of Berwald is given by [9].

$$\mathcal{B}_m R^i_{jkh} + \mathcal{B}_h R^i_{jmk} + \mathcal{B}_k R^i_{jhm} + (R^r_{shm}P^i_{jkr} + R^r_{skh}P^i_{jmr} + R^r_{smk}P^i_{jhr})y^s \ = 0.$$

Using (1.9) in above equation, then using (1.14), we get

(2.11) $$\lambda_m R^i_{jkh} + \lambda_h R^i_{jmk} + \lambda_k R^i_{jhm} + H^r_{hm}P^i_{jkr} + H^r_{kh}P^i_{jmr} + H^r_{mk}P^i_{jhr}$$
$$+\mu_m(\delta^i_j g_{kh} - \delta^i_k g_{jh}) + \mu_h(\delta^i_j g_{mk} - \delta^i_m g_{jk}) + \mu_k(\delta^i_j g_{hm} - \delta^i_h g_{jm}) = 0.$$

Transvecting (2.11) by $y^j$, using (1.9), (1.13), (1.3b) and (1.1), we get

(2.12) $$\lambda_m H^i_{kh} + \lambda_h H^i_{mk} + \lambda_k H^i_{hm} + H^r_{hm}P^i_{kr} + H^r_{kh}P^i_{mr} + H^r_{mk}P^i_{hr}$$
$$+\mu_m(y^i g_{kh} - \delta^i_k y_h) + \mu_h(y^i g_{mk} - \delta^i_m y_k) + \mu_k(y^i g_{hm} - \delta^i_h y_m) = 0.$$

Thus, we conclude

**Corollary 2.1.** *In $G(\mathcal{B}R) - RF_n$, we have the identities (2.10) and (2.12).*

## CONCLUSION

Some tensors in generalized $\mathcal{B} R -$ recurrent Finsler space have been studied. Further,certain identities belong to this space were obtained.

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