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FUZZY STRONG BI-IDEALS OF NEAR-RINGS

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ABSTRACT

In this paper we introduce the notation of fuzzy strong bi-ideal of a near-ring and obtain a characterization of a strong bi-ideal in terms of a fuzzy strong bi-ideal of a near-ring. We establish that every fuzzy left N-subgroup fuzzy left ideal of a near-ring is a fuzzy strong bi-ideal of a near-ring. But the converse is not necessarily true as shown by an example. Further, we discuss the properties of fuzzy strong bi-ideal of a near-ring.

KEYWORDS: Fuzzy two sided N-subgroup, fuzzy subnear-ring, fuzzy bi-ideal, fuzzy strong bi-ideal.

1. INTRODUCTION

The notion of fuzzy subgroup was made be Rosenfeld[9] in 1971. In,[4] W. Liu introduced the notion of fuzzy ideal of a ring. The notions of fuzzy sub near-ring, fuzzy ideal and fuzzy N-subgroup of a near-ring were introduced by Salah Abou-Zaid[11] and it has been studied by several authors.[2, 3,6-8,10-12] In this paper, we introduce the notion of a fuzzy strong bi-ideal of a near-ring and obtain the characterization of a strong bi-ideal in terms of a fuzzy strong bi-ideal of a near-ring. We establish that every fuzzy left N-subgroup or fuzzy left ideal of a near-ring is a fuzzy strong bi-ideal of a near-ring and also we establish that every left permutable fuzzy right N-subgroup or left permutable fuzzy right ideal of a near-ring is a fuzzy strong bi-ideal of a near-ring. But the converse is not necessarily true as shown by an example. Further, we discuss the properties of fuzzy strong bi-ideal of a near-ring.

2. Preliminaries

Definition: 2.1

A nonempty set N together with two binary operations "+" and "·" is called be a near-ring[1] if it satisfies the following axioms:

(i) (N,+) is a group.

(ii) (N,·) is a semi group.

(iii) $(x + y)\cdot z = (x\cdot z) + y\cdot z$, for every $x, y, z \in N$.

Note: 2.2

(i) Let X be a near-ring. Given two subsets A and B of X, $AB = \{ab / a\in A, b\in B\}$. Also we define another operation "∗" $A*B = \{a(b+i) - ab / a,b\in A, i\in B\}$.

(ii) $0x = 0$. In general $x0 \neq 0$, for some x in N.

Definition: 2.3

A near-ring N is called zero-symmetric, if $x0 = 0$, for all x in N.

Definition: 2.4

A subgroup A of (N,+) is called a bi-ideal of near-ring N if $ANA \cap (AN)*A \subseteq A$.

Definition: 2.5

An element $a\in N$ is said to be regular if for each $a\in N$, $a = aba$, for some $b\in N$

Definition: 2.6

A near-ring N is said to be left permutable near-ring if abc= bac, for all a,b,c in N.

Definition: 2.7

A function A from a non-empty set X to the unit interval [0,1]is called a fuzzy subset of N.[14]

Notation: 2.8

Let A and B be two fuzzy subsets of a semigroup N. We define the relation $\subseteq$ between A and B, the intersection and product of A and B, respectively as follows:

(i) $A \subseteq B$ if $A(x) \leq B(x)$, for all $x\in N$,

(ii) $(A\cap B)(x) = \min\{A(x), B(x)\}$, for all $x\in N$,

(iii) $(A\circ B)(x) = \begin{cases} \sup\limits_{x=yz}\{\min\{A(y), B(z)\}\} & \text{if } x = yz, \text{for all } y, z\in N, \\ 0 & \text{otherwise} \end{cases}$

It is easily verified that the "product" of fuzzy subsets is associative. Throughout this paper, N will denote a near-ring unless otherwise specified. We denote by k_I the characteristic function of a subset I of N. The characteristic function of N is denoted by N, that is, N: N $\rightarrow$ [0,1] mapping every element of N to1.

Definition: 2.9 [9]

A fuzzy subset A of a group (N,+) is said to be a fuzzy subgroup of N if for all $x,y \in N$,

(i) $A(x+y) \geq \min\{ A(x) , A(y)\}$

(ii) $A(-x) = A(x)$,

Or equivalently $A(x - y) \geq \min\{A(x) , A(y)\}$.

Note: 2.10

If A is a fuzzy subgroup of a group N, then $A(0) \geq A(x)$ for all $x \in N$.

Definition: 2.11[11]

A fuzzy subset A of N is called a fuzzy subnear-ring of N if for all $x,y \in N$,

(i) $A(x - y) \geq \min\{A(x) , A(y)\}$

(ii) $A(xy) = \min\{A(x) , A(y)\}$

Definition: 2.12[11]

A fuzzy subset A of N is said to be a fuzzy two-sided N-subgroup of N if

(i) A is a fuzzy subgroup of (N,+),

(ii) $A(xy) \geq A(x)$, for all $x,y \in N$,

(iii) $A(xy) \geq A(y)$, for all $x,y \in N$.

If A satisfies (i) and (ii), then A is called a fuzzy right N-subgroup of N. If A satisfies (i) and (iii), then A is called a fuzzy left N-subgroup of N.

Definition: 2.13[11]

A fuzzy subset A of N is said to be a fuzzy ideal of N if

(i) A is a fuzzy subnear-ring of N,

(ii) $A(y+x-y) = A(x)$, for all $x, y \in N$,

(iii) $A(xy) \geq A(x)$, for all $x, y \in N$,

(iv) $A(a(b+i) – ab) \geq A(i)$, for all $a, b, i, \in N$.

If A satisfies (i) and (ii) and (iii) then A is called a fuzzy right ideal of N. If A satisfies (i), (ii) and (iv), then A is called a fuzzy left ideal of N. In case of zero-symmetric, If A satisfies (i), (ii) and A(xy) ≥ A(y), for all x, y∈N and A is called a fuzzy left ideal of N.

3. Fuzzy Strong Bi-ideals of Near-Rings

Definition: 3.1

A fuzzy bi-ideal A of N is called a fuzzy strong bi-ideal of N, if N∘A∘A ⊆ A

Example: 3.1.1

Let N={0,a,b,c} be a near-ring with two binary operations '+' and '·' is defined as follows.

+	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

•	0	a	b	c
0	0	0	0	0
a	0	b	0	b
b	0	0	0	0
c	0	b	0	b

Define a fuzzy subset A: N →[0,1] by A(0) = 0.8, A(a) = 0.3, A(b) = 0.6, A(c) = 0.3.

Then N∘A∘A (0) = 0.3, N∘A∘A (a) = 0, N∘A∘A (b) = 0, N∘A∘A (c) = 0, and so A is a fuzzy strong bi-ideal of N.

Note: 3.2

Every fuzzy strong bi-ideal is fuzzy bi-ideal. But the converse is not true.

Example: 3.2.1

Let N={0,a,b,c} be a near-ring with two binary operations '+' and '·' is defined as follows.

+	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

•	0	a	b	c
0	0	0	0	0
a	a	a	a	a
b	0	0	b	b
c	a0	a	c	c

Define a fuzzy subset A: N →[0,1] by A(0) = 0.9, A(a) = 0.4, A(b) = 0.4, A(c) = 0.7. Then (A∘N∘A)(0) = 0.9, (A∘N∘A)(a) = 0.7, (A∘N∘A)(b) = 0.4, (A∘N∘A)(c) = 0.7, ((A∘N)∗A)(0) = 0.9 , ((A∘N)∗A)(a) = 0, ((A∘N)∗A)(b) = 0.7, ((A∘N)∗A)(c) = 0, N∘A∘A (0) = 0.3, N∘A∘A (a) = 0, N∘A∘A (b) = 0, N∘A∘A (c) = 0. Then A is a fuzzy bi-ideal of N. But not a fuzzy strong bi-ideal, since N∘A∘A (b) ⊈ A(b).

Theorem: 3.3

Let $\{A_i : i \in J\}$ be any family of fuzzy strong bi-ideals of N. Then $A = \bigcap_{i\in J} A_i$ is a fuzzy strong bi-ideal of N, where J be an index set.

Proof

By Theorem 3.4,[5] A is a fuzzy bi-ideal of N. Now for all $x \in N$, since $A = \bigcap_{i\in J} A_i \subseteq A_i$ for every $i \in J$, we have

$(\mathbf{N} \circ A \circ A)(x) \leq (\mathbf{N} \circ A_i \circ A_i)(x)$

$\leq A_i(x)$ for every $i \in J$

(since A_i is a fuzzy strong bi-ideal of N)

It follows that, $(\mathbf{N} \circ A \circ A)(x) \leq \inf\{A_i(x) : i \in J\}$

$= (\bigcap_{i\in J} A_i)(x)$

$= A(x)$

Thus $\mathbf{N} \circ A \circ A \subseteq A$. So A is a fuzzy strong bi-ideal of N.

Theorem: 3.4

Let I be a non-empty subset of N and K_I be a fuzzy subset of N. Then the following conditions are equivalent:

(i) I is a strong bi-ideal of N.

(ii) K_I is a fuzzy strong bi-ideal of N.

Proof

First assume that I is a strong bi-ideal of N. Then I is a bi-ideal of N. By Theorem 3.8,[5] we get K_I is a fuzzy bi-ideal of N.

Let a be any element of N. If $a \in I$ then $K_I(a) = 1 \geq (N \circ K_I \circ K_I)(a)$. If $a \notin I$ then $K_I(a) = 0$. On the other hand assume that $(N \circ K_I \circ K_I)(a) = 1$. Then

$$(N \circ K_I \circ K_I)(a) = \sup_{a=pq} \min\{N \circ k_I(p), k_I(q)\}$$

$$= \sup_{a=pq} \min\left\{\sup_{p=p_1p_2} \min\{\mathbf{N}(p_1), k_I(p_2)\}, k_I(q)\right\}$$

(since $N(x) = 1$, $\forall\, x \in N$)

$$= \sup_{a=pq} \min\left\{\sup_{p=p_1p_2}\{k_I(p_2)\}, k_I(q)\right\} = 1$$

and $k_I(p_2) = 1, k_I(q) = 1$. So $p_2, q \in I$. Then $a = pq = p_1 p_2 q \in NII \subseteq I$ which contradicts $a \notin I$. Thus $K_I(a) = 0 = (N \circ K_I \circ K_I)(a)$. This shows that $(N \circ K_I \circ K_I) \subseteq K_I$. Therefore K_I is a fuzzy strong bi-ideal of N.

Conversely, assume that K_I is a fuzzy strong bi-ideal of N. Every fuzzy strong bi-ideal of N is a fuzzy bi-ideal of N. Therefore by Theorem 3.8 [5], I is a fuzzy bi-ideal of N. Let a be any element of NI^2. Then there exists a, p, q, p_1 of N and the elements p_2, q of I such that a = bc and $p = p_1 p_2$.

$$(N \circ K_I \circ K_I)(a) = \sup_{a=pq} \min\{N \circ k_I(p), k_I(q)\}$$

$$= \sup_{a=pq} \min\left\{\sup_{p=p_1p_2} \min\{\mathbf{N}(p_1), k_I(p_2)\}, k_I(q)\right\}$$

$$= \sup_{a=pq} \min\left\{\sup_{p=p_1p_2} \{k_I(p_2)\}, k_I(q)\right\} = \min\{1, 1\} = 1.$$

$(K_I)(a) \geq (N \circ K_I \circ K_I)(a) = 1$. Thus $a \in I$. So $NII \subseteq I$. This shows that I is a strong bi-ideal of N.

Theorem: 3.5

Every left permutable fuzzy right N-subgroup of N is a fuzzy strong bi-ideal of N.

Proof

Let A be a left permutable fuzzy right N-subgroup of N.

To prove A is a fuzzy strong bi-ideal of N.

By Theorem 3.9,[5] we get every fuzzy right N-subgroup of N is a fuzzy bi-ideal of N. Choose $a, b, c, b_1, b_2 \in N$ such that a = bc and $b = b_1, b_2$. Then

$$N \circ A \circ A(a) = \sup_{a=bc} \min\{\mathbf{N} \circ A(b), A(c)\}$$

$$= \sup_{a=bc} \min\left\{\sup_{b=b_1b_2} \min\{\mathbf{N}(b_1), A(b_2)\}, A(c)\right\}$$

$$= \sup_{a=bc} \min\left\{\sup_{b=b_1b_2} \{A(b_2), A(c)\right\}$$

(Since A is a left permutable fuzzy right N-subgroup of N, $A(bc) = A((b_1b_2)c) = A((b_2b_1)c) > A(b_2)$) and $N(c) \geq A(c)$

$$\leq \sup_{a=bc} \min\{A(bc), \mathbf{N}(c)\}$$

$$= \sup_{a=bc} \min\{A(bc), 1\}$$

$$= \sup_{a=bc} A(bc)$$

$$= A(a)$$

Therefore $N \circ A \circ A \subseteq A$. Hence A is a fuzzy strong bi-ideal of N.

Theorem: 3.6

Every fuzzy left N-subgroup of N is a fuzzy strong bi-ideal of N.

Proof

Let A be a fuzzy left N-subgroup of N.

To prove A is a fuzzy strong bi-ideal of N.

By Theorem 3.10,[5] we get every fuzzy left N-subgroup of N is a fuzzy bi-ideal of N. Choose a, b,c, $c_1, c_2 \in$N such that a = bc and c = c_1, c_2. Then

N∘A∘A (a) = $\sup_{a=bc} \min\{\mathbf{N}(b), A \circ A(c)\}$

$= \sup_{a=bc} \min\{\mathbf{N}(b), \sup_{c=c_1c_2} \min\{A(c_1), A(c_2)\}$

$= \sup_{a=bc} \min\{1, \sup_{c=c_1c_2} \min\{A(c_1), A(c_2)\}$

(Since A is a fuzzy left N-subgroup of N, A(bc) = A(b(c_1c_2)) = A((bc_1)c_2)>A(c_2))

$\leq \sup_{a=bc} \min\{\mathbf{N}(c_1), A(bc)\}$

$= \sup_{a=bc} \min\{1, A(bc)\}$

= A(bc)

= A(a)

Therefore N∘A∘A $\subseteq$ A. Hence A is a fuzzy strong bi-ideal of N.

Theorem: 3.7

Every left permutable fuzzy two-sided N-subgroup of N is a fuzzy strong bi-ideal of N.

Proof

The proof is straight forward from the above Theorem 3.5 and Theorem 3.6

Theorem: 3.8

Every left permutable fuzzy right ideal of N is a fuzzy strong bi-ideal of N.

Proof

Let A be a left permutable fuzzy right ideal of N.

To prove A is a fuzzy strong bi-ideal of N.

By Theorem 3.12,[5] we get every fuzzy right ideal of N is a fuzzy bi-ideal of N. Choose a, b,c, $b_1, b_2 \in$N such that a = bc and b = b_1, b_2. Then

N∘A∘A (a) = $\sup_{a=bc} \min\{\mathbf{N} \circ A(b), A(c)\}$

$$= \sup_{a=bc} \min\left\{\sup_{b=b_1b_2} \min\{\mathbf{N}(b_1), A(b_2)\}, A(c)\right\}$$

$$= \sup_{a=bc} \min\left\{\sup_{b=b_1b_2} \{A(b_2), A(c)\right\}$$

(Since A is a left permutable fuzzy right ideal of N, $A(bc) = A((b_1b_2)c) = A((b_2b_1)c) > A(b_2)$) and $\mathbf{N}(c) \geq A(c)$

$$\leq \sup_{a=bc} \min\{A(bc), \mathbf{N}(c)\}$$

$$= \sup_{a=bc} \min\{A(bc), 1\}$$

$$= \sup_{a=bc} A(bc)$$

$$= A(a)$$

Therefore $N \circ A \circ A \subseteq A$. Hence A is a fuzzy strong bi-ideal of N.

Theorem: 3.9

Every fuzzy left ideal of N is a fuzzy strong bi-ideal of N.

Proof

Let A be a fuzzy left ideal of N. To prove A is a fuzzy strong bi-ideal of N.

By Theorem 3.13,[5] we get every fuzzy left ideal of N is a fuzzy bi-ideal of N. Choose a, b,c, $b_1, b_2 \in N$ such that $a = bc = b(n + c) - bn$. Then

$$\mathbf{N} \circ A \circ A\,(a) = \sup_{a=bc} \min\{\mathbf{N} \circ A(b), A(c)\}$$

$$= \sup_{a=bc} \min\left\{\sup_{b=b_1b_2} \min\{\mathbf{N}(b_1), A(b_2)\}, A(c)\right\}$$

$$= \sup_{a=bc} \min\left\{\sup_{b=b_1b_2} \{A(b_2), A(c)\right\}$$

(Since A is a fuzzy left ideal of N, $A(a) = A(bc) = A(b(n + c) - bn) > A(c)$ and $\mathbf{N}(b_2) \geq A(b_2)$)

$$\leq \sup_{a=bc} \min\{\mathbf{N}(b_2), A(b(n+c) - bn)\}$$

$$= \sup_{a=bc} A(b(n+c) - bn)$$

$$= A(bc)$$

$$= A(a)$$

Therefore $N \circ A \circ A \subseteq A$. Hence A is a fuzzy strong bi-ideal of N.

Theorem: 3.10

Every left permutable fuzzy ideal of N is a fuzzy strong bi-ideal of N.

Proof

The proof is straight forward from the Theorem 3.8 and Theorem 3.9

Remark: 3.11

The converse of Theorem 3.7 and Theorem 3.10 are not necessarily true as shown by the following example.

Example: 3.12

Let N={0,a,b,c} be the near-ring with two binary operations '+' and '•' is defined as follows.

+	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

•	0	a	b	c
0	0	0	0	0
a	0	0	0	0
b	0	0	0	a
c	00	0	0	a

Define a fuzzy subset A: N →[0,1] by A(0) = 0.75, A(a) = 0.2, A(b) = 0.3, A(c) = 0.3.

Then (A∘N∘A)(0) = 0.3, (A∘N∘A)(a) = 0, (A∘N∘A)(b) = 0, (A∘N∘A)(c) = 0, N∘A∘A (0) = 0.3, N∘A∘A (a) = 0, N∘A∘A (b) = 0, N∘A∘A (c) = 0, and so A is a fuzzy strong bi-ideal of N. Since A(a) = A(bc) $\ngeq A(b)$ and A(a) = A(bc) $\ngeq A(c)$, A is not a fuzzy two-sided N-subgroup of N. Since A(a) = A(bc) $\ngeq$ min {A(b),A(c)}, A is not a fuzzy sub near-ring of N andso A is not a fuzzy ideal of N.

Theorem: 3.13

Let A be any fuzzy strong bi-ideal of a near-ring N. Then A(axy) ≥ min{ A(x), A(y)} ∀a, x, y ∈N.

Proof

Assume that A is a fuzzy strong bi-ideal of N. Then **N**∘A∘A ⊆ A.

Let a, x and y be any element of N. Then

A(axy) ≥ (**N**∘A∘A**)** (axy)

$$= \sup_{axy=pq} \min\{\mathbf{N}\circ A(p), A(q)\}$$

$$\geq \min\{(\mathbf{N}\circ A)(ax), A(y)\}$$

$$= \min\{\sup_{ax=z_1z_2} \min\{\mathbf{N}(z_1), A(z_2)\}, A(y)\}$$

$$\geq \min\{\min\{\mathbf{N}(a), A(x)\}, A(y)\}$$

$$= \min\{\min\{1, A(x), A(y)\}$$

$= \min\{A(x), A(y)\}$

This shows that $A(axy) \geq \min\{A(x), A(y)\}$ $\forall a, x, y \in N$.

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