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WEAKNESS AND RECOVERY OF GRAEFFE'S ROOT SQUARING METHOD

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ABSTRACT

Graeffe's method is one of the roots finding method of a polynomial with real co-efficient. All the roots real and complex, repeated and non-repeated of a polynomial simultaneously are determined by Graeffe's root squaring method. In this study, some weaknesses of the Graeffe's method clearly and specifically mentioned and their recoveries have been discussed. It is said in[9] that this statement is not universally true. The method is valid if the algebraic equations satisfy the conditions, (i) equations with zero-coefficient must have at least one pair of equidistant non-zero coefficient from the zero-coefficient; (ii) any transformed equation of a given equation with non-zero coefficient may have zero coefficients but these new coefficients must satisfy (i); and (iii) all the coefficients of non-linear algebraic equation must not be unity.

KEYWORDS: Graeffe's method, root squaring, zero coefficients, equidistant, weakness, solvability conditions, transformed equation, non linear algebraic equation.

INTRODUCTION

This is a direct method to find the roots of any polynomial equation with real coefficients. According to Pan in 1997,[1] the classical problem of solving an *n*-th degree polynomial and its system has substantially influenced the development of mathematics throughout centuries.

Besides he also mentioned several important applications to the theory and practice of present-day computing of the computer age.

The n-th degree nonlinear polynomial (of a single variable x) has the form

$$a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + ... + a_1 x + a_0 = 0, \quad a_n \neq 0 \tag{1.1}$$

This type of the equation arises in many occasions as, (i) auxiliary equation of higher order ODE with constant coefficients, (ii) characteristic equations of the matrix eigenvalue problems, (iii) in the area of computer algebra and computing geometry (an active area of modern research).

General equation (1.1) has n roots (zeros), which are of following types: (i) real and distinct roots, (ii) real equal roots, (iii) imaginary or complex roots and complex roots occur in conjugate pair and (iv) combination of (i), (ii) and (iii).

Although problem of solving (1.1) was known to Sumerians (third millennium BC) for practical values of n, all the above types of roots were not known to them. After centuries this types of roots become apparent to the present generation. Now a day's many computational problems arising in the sciences, engineering, business management and statistics have been linearised and then solved by using tools from linear algebra, linear programming. Such roots involve the solution of (1.1) for smaller n. However, Computer algebra solves (1.1) for large n. There are available softwares in case of precision; error bound etc. for this causes problems and thus motivation for further research on the design of effective algorithm for solving (1.1) arises.

Thus equation (1.1) retains its major role both as a research problem and a part of practical computational tasks in the highly important area of computing called computer algebra. This information's along with others are embodied in Pan.[1]

No general algebraic method is available for the solution of equation (1.1) except for very special cases like (a) quadratic, (b) cubic and (c) quartic equation. There are some numerical procedures (methods) to determine real roots both distinct and equal only such as (i) Bisection method, (ii) False-position method, (iii) Newton-Raphson method, (iv) Secant method, (v) Muller's method, (vi) Bairstow's method, (vii) Graeffe's method of roots squaring.

Graeffe's method gives all the roots simultaneously, both real and complex. This method has not received much attention. Modern research based on computer has been carried out by Pan[1] in 1997, Malajovich[2] in 1999 and the references therein around the last part of 20th century and the beginning of 21st century. The lacks of popularity of Graeffe's method are embodied in Malajovich[2] and this article.

OBSERVATION

None of the researchers in the references discussed the following observations except a few. Their comments about the method regarding advantages and disadvantages are correct to their point of view.

(i) It is observed that there are equations, which are not transformable by root squaring into a different one with non-zero coefficients from where the roots of the equation are calculated. It is found that the odd degree equations set like,

$$x^3 + a = 0, \quad x^5 + a = 0, \quad x^7 + a = 0, \quad x^5 + 2x^4 + 2x^3 + 2x^2 + 2x + 1 = 0 \tag{2.1}$$

etc. cannot be solved by the Graeffe's root squaring method manually as well as using GRAEFFE. BAS of Constantinides.[3] All these equations (2.1) transform to the form $x^n + a = 0$, $n = 3,5,7$ after first iteration and GRAEFFE.BAS shows overflow at the statement 1080 of Constantinides.[3] It is observed that

$$\left.\begin{array}{l} x^2 + x + b = 0, \quad x^4 + a = 0, \quad x^7 + ax^2 + b = 0, \\ x^7 + ax^6 + b = 0, \quad x^4 + 2x^3 + 2x^2 + 2x + 2 = 0 \end{array}\right\} \tag{2.2}$$

etc. can be solved by both manual procedures and using program of Constantinides.[3] In this case zero coefficient revive and non zero coefficients do not vanish. As a result final transform equation in each case provides solution. Householder[4] passes a similar remark for $x^n - 1 = 0$ in which all n roots have unit moduli and the Graeffe method fails for such equations without deriving any solvability condition. Similarly, Wilkinson[5] called for equation $x^2 - 1 = 0$ well-condition zeros $x = \pm 1$, because on squaring the transformed equation becomes $x^2 - 2x + 1 = 0$ and deteriorates to coincident ill-conditioned root 1.

(ii) Actual computation reveals another fact that 4th, 6th, 8th, 10th etc. degree non-linear equations.

(iii) With positive unit coefficients like $x^4 + x^3 + x^2 + x + 1 = 0$ will never stop the procedure because the coefficients of the transformed equation remain unity *i.e.* $(-1)^4 \ f(\sqrt{x}) \ \ f(-\sqrt{x})$ remains invariant.

Weaknesses and Recoveries of Graeffe's Root Squaring Method

The most important advantages of Graeffe's root squaring method mentioned in most of the references are (i) the method gives all the roots- real, complex, equal and unequal, simple and multiple; (ii) the method needs no initial guess as in other method. Although Balagurusamy,[6] Constantinide[3] mentioned some disadvantage of coefficient growth during iteration, Malajovich and Zubelli[7] clearly and specifically point on the reasons for lack of popularity of Graeffe's method for a pretty long time. They stress two main weaknesses (i) Coefficient growth, (ii) the methods returns the module of roots but not the actual roots. Specifically reasons for Graeffe's root squaring method for lack of popularity are (i) its traditional form leads to exponents that easily exceed the maximum allowed by floating point arithmetic; (ii) Chaotic behavior of the arguments of the roots of iterates.

To overcome the weakness of (i), they introduce renormalization and to overcome weakness of (ii), they differentiate the Graeffe iteration operator, $Gf(x) = (-1)^d \ f(\sqrt{x}) \ \ f(-\sqrt{x})$ [where *d* is the degree of polynomial] and its effect is to square each root of f.[7]

Constaintinides[3] in his program alleviate serious limitation (i) above by the following 4 steps.

(a) Coefficient of the original polynomial are separated into mantissa and exponent parts and stored into two separate matrices. The ranges of mantissa used are 10.

(b) After each iteration mantissas part of coefficients are tested. If their absolute values are greater than 10 (less than 1) these values are divided by 10 (multiplied by 10) and the corresponding exponents are increased (decreased) by 1.

(c) The upper limit of exponent factor is chosen as 999 as maximum values of them may be $10 \times 10^{1.7 \times 10^{38}}$ which is not necessary for Graeffe's method. If one of the exponent factors exceeds this limit, the iteration is terminated and the roots are evaluated.

(d) Both the matrices are treated together in the program, so that exponent part of each coefficient is always accounted for.

Hazra and Loskor[8-10] observe that the algebraic equation set (2.1) is not solvable manually as well as GRAEFFE.BAS software in.[3] Thus they conclude that,

(i) equation with zero coefficient must have at least one pair of equidistant non-zero coefficient from the zero- coefficient;

(ii) any transform equation of a given equation with non-zero coefficient may have zero coefficients but these new coefficients must satisfy (i);

(iii) all the coefficients of non-linear algebraic equation must not be unity;

(iv) GRAEFFE.BAS needs modification in the light of (i), (ii) and (iii).

They suggest a testing procedure which identifies solvability of the equation. Thus software in[3] has been modified for identifying those types of problems and actually identified with the modified software.

Now, Illustration of (i): Odd degree equation of form $x^n + a = 0$ does not have a pair of equidistant non-zero coefficients from the zero coefficients as is seen from first three equations in (2.1); whereas even degree equation of the form $x^n + a = 0$ does have a pair of equidistant non-zero coefficient from the zero coefficients as is seen from 2nd equation $x^4 + a = 0$ in (2.2). Also the 3rd and 4th equations in (2.2) possess a pair of non-zero coefficient from the zero coefficient. Hence, these equations in (2.2) are solvable by both manually and using GRAEFFE.BAS software. The last equation in (2.1) has no zero coefficients but has the coefficient $a_0 = a_n = 1$ and other coefficients are equal. This type transforms to one with zero coefficients and possesses no equidistant pair of non-zero coefficients from the zero coefficients. But in the last equation of (2.2) with $a_0 \neq a_n$ and other coefficients being equal transforms into one with zero coefficients having at least a pair of non-zero coefficients from the zero coefficients. Hence this type is solvable by Graeffe's method.

In other words, all the equations in the set (2.1) reduce to the form $x^n + 1 = 0$ on the first iteration $f_1(x) = f_0(\sqrt{x})\ \ f_0(-\sqrt{x})$ and the zero- coefficient of the new transformed equation does not have a pair of non-zero coefficient from the zero coefficients. That set (2.2) does not have the above discussed problem and is solvable by Graeffe's root squaring method by both manually and using GRAEFFE.BAS software.

Modification of GRAEFFE.BAS

It is observed that there are equations, which are not transformable by root squaring into a new one with non-zero coefficients from where the roots of the original equation are calculated. It is found that the equations like

$$x^3 + a = 0, \quad x^5 + a = 0, \quad x^7 + a = 0, \quad x^5 + 2x^4 + 2x^3 + 2x^2 + 2x + 1 = 0 \qquad (3.1)$$

while running the problems with original program[3] shows "overflow" at statement 1080: X (NR) = (A(R, K-1) /A(R, K)) ^ (1/M) * 10 ^ ((FACT(R, K-1) - FACT(R, K)) / M). Overcoming ways of this difficulty have been discussed in([9] and [10]). For this purpose, the original program[3] needs modification and extension. Therefore, it should be introduced the following statements and to generate a new subroutine 7 in [3]. Statement numbers in [3] are 10, 20, 30,.......,6530 and modify statement numbers are set in-between. The following modify statements numbers are different from the original numbers. The new subroutine 7 is the extension part of the title of this paper which follows next.

```
603: IF ABS (A(R, I)) = 0 THEN GOTO 605 ELSE GOTO 610
605: IF ABS (A(R, I-1)) <> 0 AND ABS (A(R, I+1)) <> 0 THEN GOTO 1350
1075: IF C$(N-1) <> PS$ AND C$(1) <> PS$ THEN GOTO 1110
1350: GOSUB 7000
4065: IF C$(K) <> PS$ AND C$(K) <> NS$ THEN GOTO 4170
5085: IF C$(K) <> PS$ AND C$(K) <> NS$ THEN GOTO 5230
          SUBROUTINE 7
7010:   FOR I = N-1 TO 1 STEP -1
7020:   IF ABS (A(R, I)) = 0 THEN GOTO 7030
7030:   IF ABS (A(R, I-1)) <> 0 AND ABS (A(R, I+1)) <> 0 THEN GOTO 7050
7040:   NEXT I
7050:   FOR K = 1 TO N
7060:   IF X (K) + XI (K) = 0 THEN GOTO 7090
7070:   IF X (K) + XI (K) <> 0 THEN GOTO 1350
7080:   NEXT K
7090:   PRINT: PRINT   "EQUATION IS NOT SOLVABLE BY GRAEFFE'S ROOT
          SQUARING METHOD"
7100: ,
```

The difficulty of the problems (3.1) coefficients C$(N-1) and C$(1) do not show PURE SQUARE. If C$(1) do not show PURE SQUARE then from statement 1070 and statement 1110, we obtain K=0, which out of range of K at statement 1050. So we introduce new statement 1075 among them as follows:

```
1050:  FOR K = N TO 1 STEP -1
1070:  IF C$(K-1) <> PS$ THEN K = K-1: GOTO 1110
1074:  New statement
1075:  IF C$(N-1) <> PS$ AND C$(1) <> PS$: GOTO 1110
1110:  NEXT K
```

Also some of the problems (3.1) such as $x^3 + a = 0$ do not show PURESQUARE and NONSQUARE when K is N-1 to 1. In the original program it shows "overflow" at the statements 4070 and 5090. So we introduce statement 4065 and statement 5085 among them as follows:

```
4065: IF C$(K) <> PS$ AND C$(K) <> NS$: GOTO 4170
4070: R(K) = (A(R, K-1)/A(R, K+1)) ^ (1/M)*10 ^ ((FACT(R, K-1) - FACT(R, K+1))/M
5085: IF C$(K) <> PS$ AND C$(K) <> NS$: GOTO 5230
5090: R(K) = (ABS(A(R,K-1)/A(R, K+1)))^(1/M)*10^((FACT(R,K-1)-FACT(R,K+1))/M
```

The program [3] will be modified identifying the inability of solving some of the non-linear algebraic equations of the form (3.1). The modified GRAEFFE.BAS is developed using the following algorithm.

Algorithm

Identify (the presence of any zero coefficients)
or (all the coefficients are unity)
if (there is no pair of non zero coefficients from the zero coefficients)
or (all the coefficients are unity)
then go to print "equation is not solvable by Graeffe's root squaring method" Repeat the above steps for each subsequent new transformed coefficient

Numerical Examples

Some different numerical results are given below:

Example 1:

****************GRAEFFE'S ROOT-SQUARING METHOD***************

****************** (GRAEFFE.BAS) ****************

DEGREE OF POLYNOMIAL 5

COEFFICIENT 5 IS 1

COEFFICIENT 4 IS -5

COEFFICIENT 3 IS -15

COEFFICIENT 2 IS 85

COEFFICIENT 1 IS -26

COEFFICIENT 0 IS -120

GIVE THE CONVERGENCE VALUE OF F = 0.002

ROOT SQUARING PROCESS												
R	A		A		A		A		A		A	
	r	5	r	4	r	3	r	2	r	1	r	0
0	1.000E	0	-5.000E	0	-1.500E	1	8.500E	1	-2.600E	1	-1.200E	2
1	10.000E	-1	55.000E	0	10.230E	2	76.450E	2	21.076E	3	14.400E	3
2	10.000E	-1	97.900E	1	24.773E	4	16.909E	6	22.402E	7	20.736E	7
3	10.000E	-1	46.298E	4	28.712E	9	17.531E	13	43.173E	15	42.998E	15
4	10.000E	-1	15.693E	10	66.212E	19	28.254E	27	18.489E	32	18.488E	32
5	10.000E	-1	23.302E	21	42.954E	40	79.587E	55	34.182E	65	34.182E	65
6	10.000E	-1	54.210E	43	18.447E	82	63.340E	112	11.684E	132	11.684E	132
7	10.000E	-1	29.388E	88	34.028E	165	40.119E	226	13.652E	265	13.652E	265

THE COEFFICIENTS ARE

PURE SQUARE, PURE SQUARE, PURE SQUARE, PURE SQUARE, PURE SQUARE, PURE SQUARE

THE NUMBER OF SQUARING: r = 7 THE POWER: m = 128

CALCULATION OF ROOTS:

FUNCTION WITH POSITIVE VALUE OF (5) = 1.602173E-04

FUNCTION WITH NEGATIVE VALUE OF (5) = -2240.001

FUNCTION WITH POSITIVE VALUE OF (4) = -80.00003

FUNCTION WITH NEGATIVE VALUE OF (4) = 1.959801E-04

FUNCTION WITH POSITIVE VALUE OF (3) = 1.525879E-05

FUNCTION WITH NEGATIVE VALUE OF (3) = 480

FUNCTION WITH POSITIVE VALUE OF (2) = 1.478195E-05

FUNCTION WITH NEGATIVE VALUE OF (2) = 280.0001

FUNCTION WITH POSITIVE VALUE OF (1) = -80

FUNCTION WITH NEGATIVE VALUE OF (1) = -8.583069E-06

THE 5 ROOTS ARE:

5

-4

3

2

-1

Example 2:

****************GRAEFFE'S ROOT-SQUARING METHOD****************

***************** (GRAEFFE.BAS) ****************

DEGREE OF POLYNOMIAL 5

COEFFICIENT 5 IS 1

COEFFICIENT 4 IS -10

COEFFICIENT 3 IS 42

COEFFICIENT 2 IS -102

COEFFICIENT 1 IS 145

COEFFICIENT 0 IS -100

GIVE THE CONVERGENCE VALUE OF F = 0.001

ROOT SQUARING PROCESS												
R	A		A		A		A		A		A	
	r	5	r	4	r	3	r	2	r	1	r	0
0	1.000E	0	-1.000E	1	4.200E	1	1.020E	2	1.450E	2	-1.000E	2
1	10.000E	-1	1.600E	1	1.400E	1	0.224E	3	0.625E	3	0.000E	3
2	10.000E	-1	22.800E	1	-57.220E	2	3.527E	5	-40.894E	5	10.000E	7
3	10.000E	-1	63.428E	3	-13.626E	7	12.318E	10	-53.812E	12	10.000E	15
4	10.000E	-1	42.957E	8	28.322E	14	17.776E	20	43.212E	25	10.000E	31
5	10.000E	-1	18.447E	18	-72.495E	29	15.712E	41	-16.879E	52	10.000E	63
6	10.000E	-1	34.029E	37	-54.139E	59	39.046E	82	-29.349E	104	10.000E	127
7	10.000E	-1	11.580E	76	-23.643E	121	18.874E	166	-69.479E	210	10.000E	255
8	10.000E	-1	13.409E	153	12.188E	243	50.851E	332	10.525E	422	10.000E	511
9	10.000E	-1	17.981E	307	12.183E	486	28.833E	665	90.741E	843	0.000E	1023

FACTOR EXCEEDS 999. POSSIBILITY OF COMPLEX OR REPEATED ROOTS.

THE COEFFICIENTS ARE

PURE SQUARE, PURE SQUARE, NON SQUARE, NON SQUARE, NON SQUARE, PURE SQUARE

THE NUMBER OF SQUARING: r = 8 THE POWER: m = 256

CALCULATION OF ROOTS:

FUNCTION WITH POSITIVE VALUE OF (4.000002) = 1.125336E-04 FUNCTION WITH NEGATIVE VALUE OF (4.000002) = -8584.013

TWO PAIRS OF COMPLEX OR REPEATED ROOTS WITH IDENTICAL MODULI:

CHECK FOR REPEATED ROOTS:

FUNCTION WITH POSITIVE VALUE OF (2.236068) = -10.29418 FUNCTION WITH NEGATIVE VALUE OF (2.236068) = -1709.705

WARNING: CONVERGENCE NOT SATISFIED BY REAL ROOT* *ROOTS ARE COMPEX*

*THE 5 ROOTS ARE:

4

2+1i

2-1i

1+2i

1-2i

Example 3:

****************GRAEFFE'S ROOT-SQUARING METHOD****************

****************** (GRAEFFE.BAS) ****************

DEGREE OF POLYNOMIAL 5

COEFFICIENT	5	IS	1
COEFFICIENT	4	IS	0
COEFFICIENT	3	IS	0
COEFFICIENT	2	IS	0
COEFFICIENT	1	IS	0
COEFFICIENT	0	IS	1

GIVE THE CONVERGENCE VALUE OF F = 0.001

ROOT SQUARING PROCESS												
R	A		A		A		A		A		A	
	R	5	r	4	r	3	r	2	r	1	r	0
0	1.000E	0	0.000E	0	0.000E	0	0.000E	0	0.000E	0	1.000E	0
1	10.000E	-1	0.000E	-1	0.000E	-1	0.000E	-1	0.000E	-1	10.000E	-1
2	10.000E	-1	0.000E	-3	0.000E	-3	0.000E	-3	0.000E	-3	10.000E	-1
3	10.000E	-1	0.000E	-7	0.000E	-7	0.000E	-7	0.000E	-7	10.000E	-1
4	10.000E	-1	0.000E	-15	0.000E	-15	0.000E	-15	0.000E	-15	10.000E	-1
5	10.000E	-1	0.000E	-31	0.000E	-31	0.000E	-31	0.000E	-31	10.000E	-1
6	10.000E	-1	0.000E	-63	0.000E	-63	0.000E	-63	0.000E	-63	10.000E	-1
7	10.000E	-1	0.000E	-127	0.000E	-127	0.000E	-127	0.000E	-127	10.000E	-1
8	10.000E	-1	0.000E	-255	0.000E	-255	0.000E	-255	0.000E	-255	10.000E	-1
9	10.000E	-1	0.000E	-511	0.000E	-511	0.000E	-511	0.000E	-511	10.000E	-1
10	10.000E	-1	0.000E	%-1023								

FACTOR EXCEEDS 999. POSSIBILITY OF COMPLEX OR REPEATED ROOTS.

THE COEFFICIENTS ARE:

PURE SQUARE, , , , ,PURE SQUARE

THE NUMBER OF SQUARING: r = 9 THE POWER: m = 512

CALCULATION OF ROOTS:

THE 5 ROOTS ARE:

0

0

0

0

0

EQUATION IS NOT SOLVABLE BY GRAEFFE’S ROOT SQUARING METHOD

CONCLUSION

Literature survey of the past and the recent has been discussed about the Classical method of Graeffe’s root squaring method. Graeffe's method has a number of drawbacks. Some problems are not solvable manually and using GRAEFFE. BAS software [3] which has been discovered under observations and this software modified and extended in [9]. Besides, in this paper the lack of popularity, weaknesses of the method and their recoveries have been discussed.

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