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PROPERTIES OF PERIODIC PULSE SIGNALS

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ABSTRACT

This paper considers properties of periodic single pulse signals. The primary property of periodic single pulse signals is a perfect periodic autocorrelation function. Beside of this property, periodic single pulse signals can be presented as a linear composition of orthogonal periodic pulse signals with zero cross correlation. This paper focuses on the properties of these orthogonal periodic pulse signals and performs a frequency interpretation of its presentation. The results of the study can be applied to any electronic periodic pulse signal systems including radar, radio navigation, communication, and telemetric systems.

KEYWORDS: periodic single pulse signals (PSPS), orthogonal signals, perfect periodic autocorrelation function (ACF), cross correlation function (CCF), radar, radio navigation and communication systems.

INTRODUCTION

This paper is a continuation of reference [1] which introduced periodic single pulse signals as a linear composition of orthogonal periodic pulse signals. Periodic single pulse signals (PSPS) with a rectangular pulse duration of T and time repetition $T_{rep} = NT$ (Fig. 1a) can be presented as

$$S_{NT}(t) = N \cdot S^{(1)}{}_{NT}(t), \qquad (1)$$

Where N (N > 1) are positive integers. Signals $S^{(1)}_{NT}$ (t) are periodic unit pulse signals, i.e. periodic rectangular single pulse signals of duration T, with the magnitude of each pulse equal to 1, and with a period of repetition NT. For signals S_{NT} (t) (1), the magnitude N of each pulse is related to the time repetition T_{rep} = NT (Fig. 1a). This relation was made in order to simplify subsequent calculations.

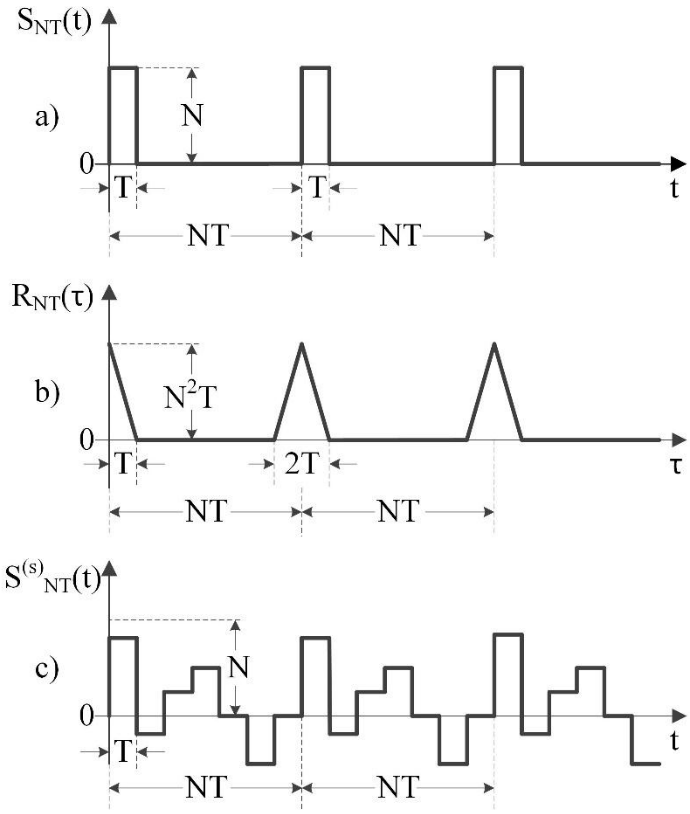

Figure 1: Signals S_{NT} (t) and $S^{(s)}_{NT}$ (t).

These types of PSPS (1) (Fig. 1a) are well known and widely used in electronic systems.[4],[5],[6],[7] Many electronic telemetric, communication and radar systems utilize these signals since the properties of PSPS are well understood in the time and frequency domains. The primary useful property of PSPS in the time domain is its perfect periodic autocorrelation function (ACF) R_{NT} (τ) which has only one maximum (without any side lobes) on time interval NT (Fig. 1b). The perfect ACF of PSPS combined with the simplicity of generation and processing the signals makes use of PSPS very convenient in radar, telemetric and communication systems for synchronization, time delay measurements, etc. purposes.

It was shown in reference[1] that for any N (N > 1) (1) which are not prime numbers, $N \neq N_p$ ($N_p = 2, 3, 5, 7, 11$, etc.), signals S_{NT} (t) (PSPS) (1) (Fig. 1a) can be presented as a linear composition of periodic orthogonal pulse signals with zero cross correlation. Using this presentation, it is possible to transform PSPS into periodic time spread pulse signals (PTSPS) $S^{(s)}_{NT}$ (t) (Fig 1c). During this transformation procedure, the energy of PSPS, which was concentrated on narrow interval T, is spread over the entire interval NT, i.e. all over the time of repetition T_{rep} = NT (Fig. 1c). Both signals (PSPS) S_{NT} (t) (Fig. 1a) and (PTSPS) $S^{(s)}_{NT}$ (t) (Fig. 1c) have the same perfect periodic autocorrelation function (Fig. 1b).

This paper considers a more general approach for representing PSPS and transforming them into PTSPS. As mentioned above, reference[1] introduced PSPS representations and the process of transforming them into PTSPS for integers N (N > 1), where N are not prime numbers ($N \neq N_p$). In this paper, it is shown that for any integers N (N > 1), including prime numbers ($N = N_p$), PSPS S_{NT} (t) (1) can be presented as a composition of periodic orthogonal pulse signals with zero cross correlation, and then PSPS can be transformed into PTSPS. Below, we perform a frequency interpretation of this procedure, and discuss several properties of periodic orthogonal pulse signals with zero cross correlation.

PSPS as Linear Composition of Periodic Orthogonal Pulse Signals

Any N (N > 1) (1), where N are positive integers (including prime numbers, $N = N_p$), can be presented as the product

$$N = \prod_{i=1}^{n} N_i = N_1 \cdot N_2 \cdot \ldots \cdot N_n = 1 \cdot N_2 \cdot \ldots \cdot N_n, \qquad (2)$$

Where N_i = 1, for $i = 1$, and N_i (N_i > 1) are positive integers for i = 2, 3, 4, …., n. Notice, that the value of variable i in (2) is only related to the order of the multipliers (factors). For instance, the value N = 30 can be presented as products in the following ways: 1) $N = 1 \cdot 2 \cdot 3 \cdot 5$, 2) $N = 1 \cdot 3 \cdot 2 \cdot 5$, and 3) $N = 1 \cdot 5 \cdot 2 \cdot 3$; and the values of N_1 = 1 (i = 1) are the same for all these three cases. But the values of N_2 (i = 2) are different in all of these three cases. The N_2 values are N_2 = 2, N_2 = 3, and N_2 = 5 for case 1, case 2, and case 3 respectively. Notice that the first multiplier in (2), i.e. multiplier N_1 (i = 1), always has a value of 1.

After presenting N as a product (2), periodic single pulse signals (1) can be presented as a composition of n independent orthogonal periodic pulse signals with zero cross correlation:

$$S_{NT}(t) = N \cdot S^{(1)}_{NT}(t) = \sum_{i} P_{Ni}(t), \qquad (3)$$

Where i = 1, 2, 3, …, n (2), and P_{Ni} (t) are periodic pulse signals of duration T with zero cross correlation. The characteristics of periodic pulse signals P_{Ni} (t) are defined by the values of N_i (2).[1]

The product (2) introduces a multiplier with value 1, namely $N_1 = 1$, to (1) as a new feature. This new multiplier does not change the resulting product, but by using this multiplier it is possible (based on the definition of prime numbers) to present prime numbers Np as a product of two values

$$N_p = \prod_{i=1}^{2} N_i = N_1 \cdot N_2 = 1 \cdot N_p, \qquad (4)$$

and then present PSPS S_{NT} (t) (1), where $N = N_p$, as the sum of two independent orthogonal periodic pulse signals with zero cross correlation (3).

Notice, that any positive integers N (N > 1), including prime numbers $N = N_p$, can also be presented by a product of only two multipliers

$$N = \prod_{i} N_i = N_1 \cdot N_2 = 1 \cdot N_2 = 1 \cdot N, \qquad (5)$$

Where i =1, 2 (n = 2), $N_1 = 1$, and $N_2 = N$.

After presentation (2) as a product of only two multipliers (5), any periodic single pulse signals with period of repetition NT (1) can also be presented as a composition of only two orthogonal periodic pulse signals (2), (3) and (5) with zero cross correlation (Fig. 2)

$$S_{NT}(t) = N \cdot S^{(1)}{}_{NT}(t) = \sum_{i} P_{Ni}(t) = P_{N1}(t) + P_{N2}(t), \quad i = 1, 2 \qquad (6)$$

Where P_{N1} (t) and P_{N2}(t) are independent, i.e. non-correlated, periodic pulse signals with zero cross correlation corresponding to multipliers $N_1 = 1$ and N_2 ($N_2 > 1$) (5) respectively.

The main characteristics[1] of periodic pulse signals P_{N1} (t) and P_{N2} (t) are magnitudes p_{N1} and p_{N2}, and the periods of repetitions T_1 and T_2 (Fig. 2b and Fig. 2c correspondingly). In accordance with the rules defined in[1], the periodic pulse signal P_{N1} (t) has a magnitude $p_{N1} = N_1 = 1$ and a period of repetition $T_1 = N_1 \cdot T = T$ (Fig. 2b). Notice that the period of repetition of P_{N1} (t) equals the length of the pulse signal, i.e. $T_1 = T$, and thus periodic pulse signal P_{N1} (t) is a direct current (DC) signal with magnitude $p_{N1} = N_1 = 1$ consisting of rectangular pulses of duration T with period of repetition also equals to T (Fig. 2b). This signal corresponds to a signal with zero frequency. The periodic pulse signal P_{N2} (t) has a magnitude $p_{N2} = N_1 \cdot (N_2 - 1) = 1 \cdot (N-1) = (N-1)$ and a period of repetition $T_2 = (N_1 \cdot N_2) \cdot T = N \cdot T$[1] (Fig. 2c).

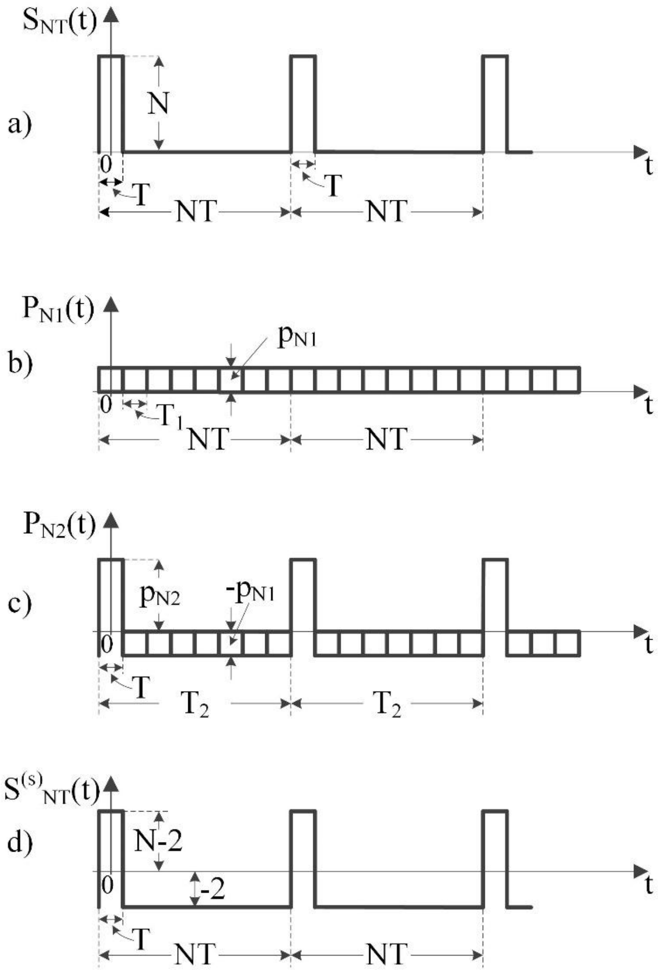

Figure 2: Signals S_{NT} (t), P_{N1} (t), P_{N2} (t) and $S^{(s)}$ (t) for $i = 1,2$ (n = 2).

It is more convenient to present periodic pulse signals as numeric sequences such as: S_{NT} (t) ⇔ S_N, where $S_N = \{s\} = (s_1, s_2, \ldots, s_N)$ and P_{Ni} (t) ⇔ P_{Ni}.[1], [3] Applying this presentation to Figure 2 where i =1, 2 (n = 2) and $N = N_1 \cdot N_2 = 1 \cdot N$, we obtain

$$\begin{array}{lll} i = 1 & P_{N1} = (p_1, \ p_1, \ldots\ldots, \ \ p_1, \ p_1) = (1, \quad 1, \quad .., 1, 1) & \\ i = 2 & P_{N2} = (p_2, -p_1, \ldots\ldots, -p_1, -p_1) = ((N\text{-}1), -1, \ldots\ldots,-1,-1) & (7) \\ & \text{---} & \\ & S_N \ = \quad P_{N1} + P_{N2} \quad = (N, \quad 0, \ldots\ldots, 0, 0) & \end{array}$$

Where $S_N = P_{N1} + P_{N2}$, $p_1 = N_1$, $p_2 = N_1 \cdot (N_2 - 1)$ and $N = N_1 \cdot N_2 = 1 \cdot N$; i.e. $N_1 = 1$, $N_2 = N$, $p_1 = 1$, $p_2 = N - 1$, and $S_N = (N, 0, 0, \ldots, 0, 0)$ (Fig. 2a).

As shown in[1], each different combination of multipliers in (2) correspond to a different combination of periodic pulse signals P_{Ni} (t) in the presentation of PSPS (3). Because N_i must equal 1 when $i = 1$, for integers $N = N_p$ equation (2) can be presented as a combination of only two multipliers, namely $N_1 = 1$ and $N_2 = N_p$ (5); i.e. in this case equation (2) has only one combination of multipliers. This result means that for integers N which are prime numbers

($N = N_p$), it is possible to obtain only one presentation of PSPS which corresponds to sequences (5), (6) and (7) when $N = N_p$.

In the case when integers N are not prime numbers ($N \neq N_p$), equation (2) can be presented as a combination of more than two multipliers. As a result of this presentation, it is possible to obtain more than one presentation of PSPS (3) for the same value of N, depending on different combinations of multipliers in (2).

For instance, these are the eight different combinations of multipliers for $N = \prod N_i = 12$

$$\begin{aligned}
&1.\ N = N_1 \cdot N_2 = 1 \cdot 12 = 12\\
&2.\ N = N_1 \cdot N_2 \cdot N_3 \cdot N_4 = 1 \cdot 2 \cdot 2 \cdot 3 = 12\\
&3.\ N = N_1 \cdot N_2 \cdot N_3 \cdot N_4 = 1 \cdot 2 \cdot 3 \cdot 2 = 12\\
&4.\ N = N_1 \cdot N_2 \cdot N_3 \cdot N_4 = 1 \cdot 3 \cdot 2 \cdot 2 = 12\\
&5.\ N = N_1 \cdot N_2 \cdot N_3 = 1 \cdot 4 \cdot 3 = 12\\
&6.\ N = N_1 \cdot N_2 \cdot N_3 = 1 \cdot 3 \cdot 4 = 12\\
&7.\ N = N_1 \cdot N_2 \cdot N_3 = 1 \cdot 2 \cdot 6 = 12\\
&8.\ N = N_1 \cdot N_2 \cdot N_3 = 1 \cdot 6 \cdot 2 = 12.
\end{aligned} \tag{8}$$

Each combination of multipliers for N =12 (8) corresponds different combination of periodic pulse signals $P_{Ni}(t)$ in the presentation of PSPS (3).[1]

The following samples present PSPS (1) as a composition of $P_{Ni}(t)$ signals with zero cross correlation for N = 12 (8) using (2) and (3):

1. $N = N_1 \cdot N_2 = 1 \cdot 12 = 12$: $N_1 = 1, N_2 = 12, i = 1, 2$ (n = 2)

$$\begin{aligned}
&i=1 \quad P_{N1} = (p_1,\ p_1, \ldots\ldots,\ p_1,\ p_1) = (1,\ 1, 1, 1, 1, 1,\ 1,\ 1, 1, 1, 1, 1)\\
&i=2 \quad P_{N2} = (p_2, -p_1, \ldots\ldots, -p_1, -p_1) = (11,-1,-1,-1,-1,-1,-1,-1,-1,-1,-1,-1)\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&\qquad S_N = \quad P_{N1} + P_{N1} \quad = (12, 0, \ldots\ldots\ldots\ldots\ldots\ldots, 0, 0),
\end{aligned} \tag{9}$$

Where $S_N = P_{N1} + P_{N2}$, $p_1 = N_1 = 1$, $p_2 = N_1 \cdot (N_2 - 1) = 11$.

2. $N = N_1 \cdot N_2 \cdot N_3 = 1 \cdot 3 \cdot 4 = 12$: $N_1 = 1, N_2 = 3, N_3 = 4, i = 1, 2, 3$ (n = 3)

$$\begin{aligned}
&i=1 \quad P_{N1} = (p_1, p_1, \ldots\ldots,\ p_1, p_1) = (1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1)\\
&i=2 \quad P_{N2} = (p_2, -p_1, \ldots\ldots, -p_1, -p_1) = (2,-1,-1, 2,-1,-1, 2,-1,-1, 2,-1,-1)\\
&i=3 \quad P_{N3} = (p_3, 0, \ldots\ldots\ldots\ldots, 0) = (9, 0, 0, -3, 0, 0, -3, 0, 0, -3, 0, 0)\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&\qquad S_N = (P_{N1} + P_{N2} + P_{N3}) \quad = (12, 0, \ldots\ldots\ldots\ldots\ldots,\ 0, 0, 0),
\end{aligned} \tag{10}$$

Where $S_N = P_{N1} + P_{N2} + P_{N3}$, $p_1 = N_1 = 1$, $p_2 = N_1 \cdot (N_2 - 1) = 2$, and $p_3 = N_1 \cdot N_2 \cdot (N_3 - 1) = 9$.

3. $N = N_1 \cdot N_2 \cdot N_3 = 1 \cdot 4 \cdot 3 = 12$, $N_1 = 1, N_2 = 4, N_3 = 3, i = 1, 2, 3$ (n = 3)

$$\begin{aligned}
&i=1 \quad P_{N1} = (p_1,\ p_1, \ldots\ldots,\ p_1,\ p_1) = (1, 1, 1,\ 1, 1, 1,\ 1, 1,\ 1, 1, 1, 1)\\
&i=2 \quad P_{N2} = (p_2, -p_1, \ldots\ldots, -p_1, -p_1) = (3,-1,-1,-1, 3,-1,-1,-1,\ 3,-1,-1,-1)
\end{aligned} \tag{11}$$

i=3 P_{N3} = (p_3, 0, ..., 0) = (8, 0, 0, 0, -4, 0, 0, 0, -4, 0, 0, 0)

....

S_N = ($P_{N1} + P_{N2} + P_{N3}$) = (12,0, ………………….., 0, 0, 0),

Where $S_N = P_{N1} + P_{N2} + P_{N3}$, $p_1 = N_1 = 1$, $p_2 = N_1 \cdot (N_2 - 1) = 3$, and

$p_3 = N_1 \cdot N_2 \cdot (N_3 - 1) = 8$.

As follows from (9), (10) and (11), for all of these cases when $N = 1 \cdot 12$, $N = 1 \cdot 3 \cdot 4$, and $N = 1 \cdot 4 \cdot 3$, PSPS S_{NT} (t) can be presented as the linear composition, i.e. the sum, of orthogonal zero cross correlation signals P_{Ni} (t) (3). The number of orthogonal zero cross correlation signals P_{Ni} (t) corresponds to the number of multipliers (factors) n in multiplication N (2), (3). Notice that integers N (N > 1) are related with magnitude of single pulse signals and also related with duration of repetition period NT (1). And parameters of orthogonal zero cross correlation signals P_{Ni} (t) (9), (10) and (11) are defined by values of Ni (2), (3).[1]

Frequency Interpretation of PSPS Presentation

The frequency characteristics S_{NT} (ω) of PSPS S_{NT} (t) (Fig. 2a) are discrete functions and consist of discrete frequency components $S_{NT}(\omega_k)$, where $\omega_k = k \cdot 2\pi/NT$, k = 0, 1, 2, 3, 4, ….., and the signal repetition period is T_{rep} = NT. The discrete frequency components $S_{NT}(\omega_k)$ are defined by Fourier series[7] as

$$S_{NT}(\omega_0) = S_{NT}(t{=}0)/q = 1, \text{ for } k = 0, \qquad (12)$$

$$S_{NT}(\omega_k) = 2 \cdot S_{NT}(t{=}0)/k\pi \cdot \sin(k\pi/q)$$

$$= 2N/k\pi \cdot \sin(k\pi/N), \text{ for } k > 0 \ (k = 1, 2, 3, \ldots),$$

Where S_{NT} (t = 0) = N (Fig. 2a) is the magnitude of the rectangular single pulse, q is the ratio of repetition time T_{rep} = NT to duration of single rectangular pulse T, i.e. $q = T_{rep} / T = N$.

Table 1 represents the amplitude coefficients $S_{NT}(\omega_k)$ for N = 12, i.e. for the examples above when N = 12 (9), (10) and (11).

Table 1: Amplitude Coefficients $S_{NT}(\omega_k)$.

k	0	1	2	3	4	5	6	7	8	9	10	11	12
$S_{NT}(\omega_k)$	1	1.98	1.91	1.80	1.65	1.48	1.27	1.05	0.83	0.60	0.38	0.18	0

On Figure 3a, the amplitude frequency characteristic S_{NT} (ω) of PSPS S_{NT} (t) (Fig. 2a) for N = 12 is plotted in accordance with Table 1 on the frequency interval from ω = 0 to ω = 2π/T (k = 0, 1, 2, …, 11, 12). The frequency interval ω = 2π/T represents the duration of the main lobe of the frequency characteristic of a single rectangular pulse of duration T.

In the case of $N = 1 \cdot 12 = 12$ (9), there are only two periodic pulse signals P_{Ni} (t) that can present PSPS S_{NT} (t) (3), (4), (5) and (6), because the product (2), (4) only consists of two factors $N_1 = 1$ and $N_2 = 12$.

Figure 3b presents the frequency characteristic P_{N1} (ω) of the signal P_{N1} (t) (9) (Fig. 2b). The frequency characteristic P_{N1} (ω) corresponds to the direct current (DC) signal, i.e. the signal with frequency $\omega_k = \omega_0$, where k = 0 ($\omega = \omega_0 = 0$) (Table 1).

The frequency characteristic P_{N2} (ω) of signal P_{N2} (t) (Fig. 2c) is presented in Fig. 3c. The frequency characteristics P_{N2} (ω) have a value of zero on frequency $\omega = \omega_k = \omega_0 = 0$ (k=0). Thus, signals P_{N1} (t) and P_{N2} (t) (9) (Figs. 2b and 2c) do not have any mutual frequency components, and as the result these signals are orthogonal signals with zero cross correlation.

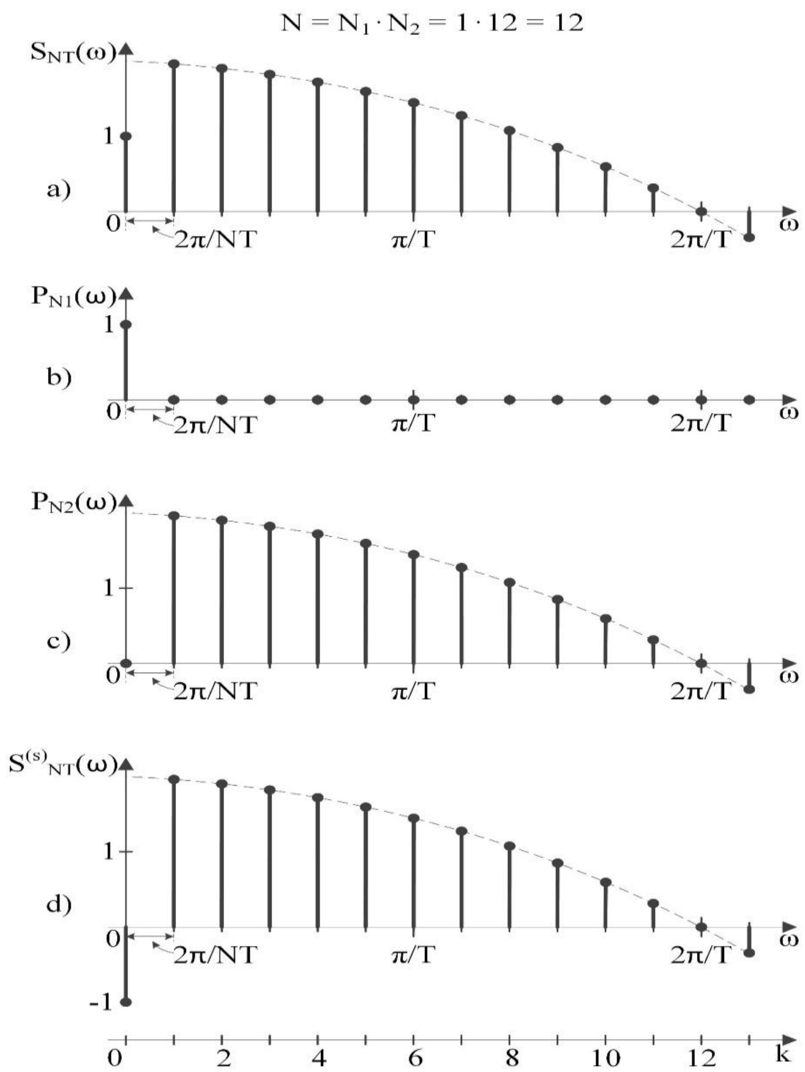

Figure 3: Frequency characteristics of signals S_{NT}(t) and $S^{(s)}_{NT}$ for $N = 1 \cdot 12 = 12$.

The case **N = 1 · 3 · 4 = 12** (10) corresponds to three periodic pulse signals P_{Ni} (t) due to the existence of three factors (2), (3) and (10). In this case, the frequency spectrum S_{NT} (ω) of signal S_{NT} (t) (10) and the frequency spectrum P_{N1} (ω) of signal P_{N1} (t) (10) (Figs. 4a and 4b correspondingly) are the same as for signal PSPS with N = 1 · 12 = 12 (9) (Fig. 3a and Fig. 3b). But the frequency spectrum of signal P_{N2} (t) is different and pulse signal P_{N3} (t) emerged.

One may easily observe (10) that periodic signal P_{N2} (t) is periodic signal S_{NT} (t) = S_{3T} (t) (N=3) (Fig. 2a) minus signal P_{N1} (t) (Fig. 2b). The frequency spectrum of signal S_{NT} (t) = S_{3T} (t) can be calculated using (12). On the frequency interval from ω = 0 to ω = 2π/T, this spectrum consists of three frequency components located at frequencies ω = 0, ω = 2π/3T, and ω = 4π/3T. These frequencies correspond to frequencies ω_k, where k = 0, k = 4, and k = 8 (Table 1), and the values of these frequency components correspond to the values of S_{NT} (ω_k) for k = 0, k = 4, and k = 8 (Table 1). Thus, the discrete spectrum of signal S_{NT} (t) = S_{3T} (t) consists of three frequency components S_{NT} (ω_k) with k = 0, k = 4, and k = 8 (Fig. 4). The signal P_{N2} (t) can be obtained by subtracting signal P_{N1} (t) from signal S_{NT} (t) = S_{3T} (t). After performing the subtraction, the frequency spectrum of signal P_{N2} (t) has zero value on frequency $\omega = \omega_k = \omega_0 = 0$ (k = 0). Correspondingly, the frequency spectrum P_{N2} (ω) of signal P_{N2} (t) will consist of only two frequency components S_{NT} (ω_k) with k = 4 and k = 8 (Fig. 4c).

One may easily observe that periodic signal P_{N3} (t) (10) consists of periodic signal S_{NT} (t) (N = 12) minus periodic signal S_{NT} (t) = S_{3T} (t) (N=3) (Fig. 2). The frequency spectrum P_{N3} (ω) can be obtained by subtracting spectrum S_{3T} (ω) of signal S_{NT} (t) = S_{3T} (t) from spectrum S_{NT} (ω) (N=12) (Fig. 4a). As shown above, spectrum S_{3T} (ω) of signal S_{NT} (t) = S_{3T} (t) consists of three frequency components S_{NT} (ω) corresponding to frequencies with k = 0, k = 4, and k = 8 (Fig. 4). After subtraction of spectrum S_{3T} (ω) from spectrum S_{NT} (ω) (N = 12), it is possible to obtain the spectrum P_{N3} (ω) of periodic signal P_{N3} (t) (Fig. 4d).

Both spectrums P_{N2} (ω) and P_{N3} (ω) (Fig. 4c and Fig. 4d) have different frequency components. Spectrum P_{N2} (ω) consists of two components, namely frequency components with k = 4 and k = 8. But spectrum P_{N3} (ω) consists of nine frequency components with k = 1, 2, 3, 5, 6, 7, 9, 10, and 11. Figure 4 shows that all these three signals P_{N1} (t), P_{N2} (t) and P_{N3} (t) (10) are orthogonal signals with zero cross correlation because all these three signals do not have mutual frequency components.

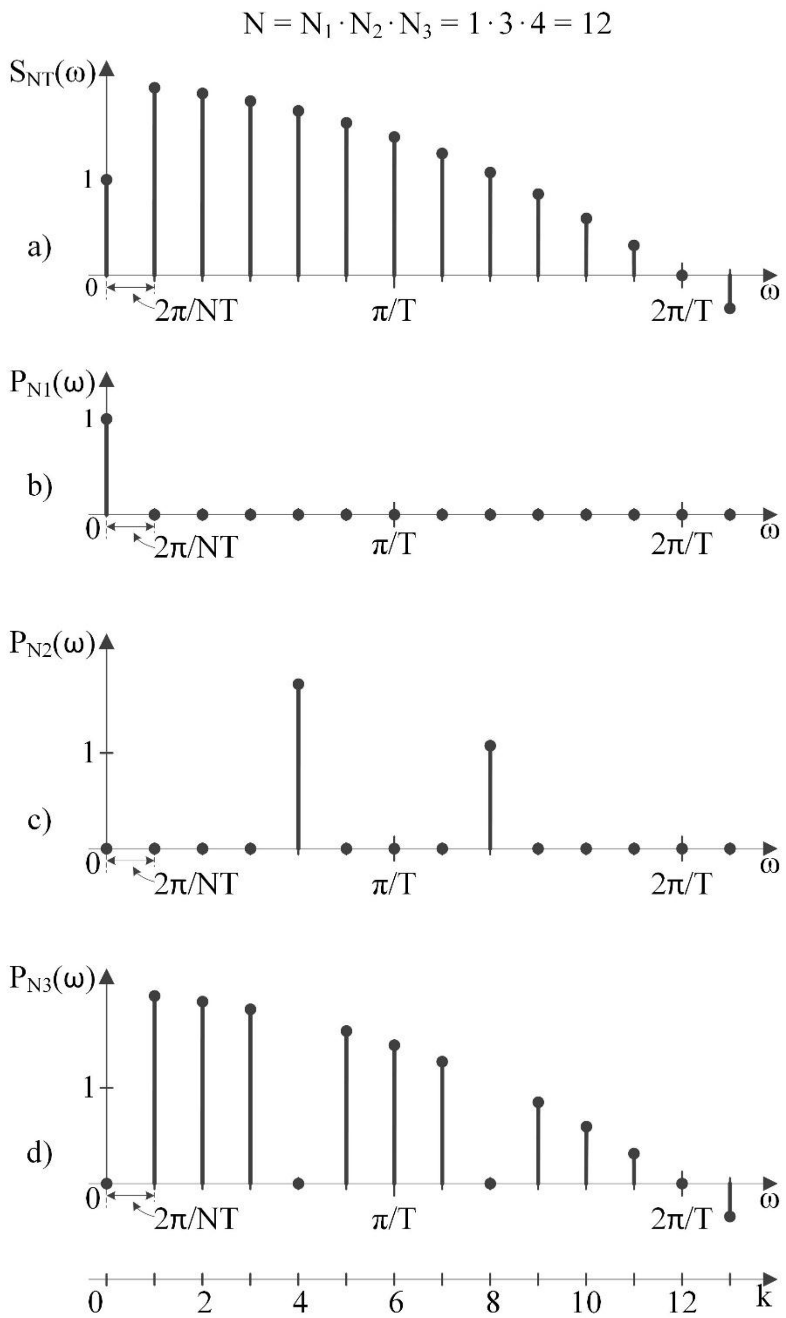

Figure 4: Frequency spectrum of signals P_{N1} (t), P_{N2} (t) and P_{N3} (t) for N = 1 · 3 · 4 = 12.

The case N = 1 · 4 · 3 = 12 (11) corresponds to three periodic pulse signals P_{Ni} (t) due to the existence of three factors (2), (3) and (11). As in the previous case, the frequency spectrum S_{NT} (ω) of signal S_{NT} (t) (11) and the frequency spectrum P_{N1} (ω) of signal P_{N1} (t) (11) (Fig. 5a and Fig. 5b correspondingly) are the same as for the signal PSPS with N = 1 · 12 = 12 (9) (Fig. 3a and Fig. 3b). But the frequency spectrum of signal P_{N2} (t) is different and pulse signal P_{N3} (t) emerged.

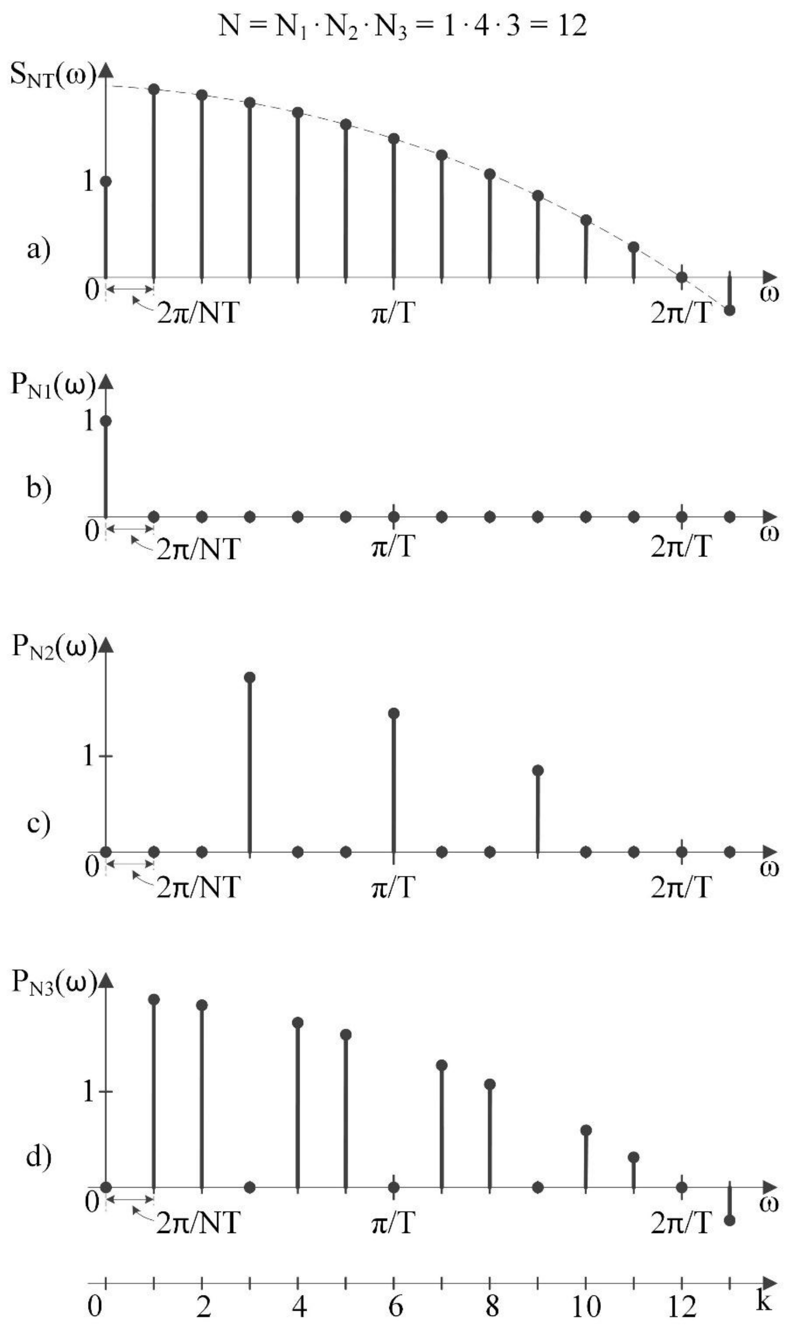

Figure 5: Frequency spectrum of signals P_{N1} (t), P_{N2} (t) and P_{N3} (t) for N = 1 · 4 · 3 = 12.

One may easily observe (11) that periodic signal P_{N2} (t) is periodic signal S_{NT} (t) = S_{4T} (t) (N=4) (Fig. 2a) minus signal P_{N1} (t) (Fig. 2b). The frequency spectrum of signal S_{NT} (t) = S_{4T} (t) can be calculated using (12). On the frequency interval from ω = 0 to ω = 2π/T, this spectrum consists of four frequency components located at frequencies ω = 0, ω = 2π/4T, ω = 4π/4T, and ω = 6π/4T. These frequencies correspond to frequencies ω_k, where k = 0, k = 3, k = 6, and k = 9 (Table 1), and the values of these frequency components correspond to values of S_{NT} (ω_k) with k = 0, k = 3, k = 6, and k = 9 (Table 1). Thus, the discrete spectrum of signal

$S_{NT}(t) = S_{4T}(t)$ consists of four frequency components $S_{NT}(\omega_k)$ for $k = 0$, $k = 3$, $k = 6$, and $k = 9$ (Fig. 5). The signal $P_{N2}(t)$ can be obtained by subtracting signal $P_{N1}(t)$ from signal $S_{NT}(t) = S_{4T}(t)$. After the subtraction procedure signal $P_{N2}(t)$ has zero value on frequency $\omega = \omega_k = \omega_0 = 0$ (k=0). And correspondingly, the frequency spectrum of signal $P_{N2}(t)$ will consist of only three frequency components $S_{NT}(\omega_k)$ with $k = 3$, $k = 6$, and $k = 9$ (Fig. 5c).

One can observe that periodic signal $P_{N3}(t)$ (11) consists of periodic signal $S_{NT}(t)$ (N = 12) minus periodic signal $S_{NT}(t) = S_{4T}(t)$ (for N = 4) (Fig. 2). The frequency spectrum $P_{N3}(\omega)$ of periodic signal $P_{N3}(t)$ can be obtained by subtracting spectrum $S_{4T}(\omega)$ of signal $S_{NT}(t) = S_{4T}(t)$ from spectrum $S_{NT}(\omega)$ (N=12) (Fig. 5a). As above, spectrum $S_{4N}(\omega)$ of signal $S_{NT}(t) = S_{4T}(t)$ consists of four frequency components $S_{NT}(\omega)$ (N=12) corresponding to frequencies with $k = 0$, $k = 3$, $k = 6$, and $k = 9$ (Fig. 5). After subtracting spectrum $S_{4T}(\omega)$ from spectrum $S_{NT}(\omega)$ (N=12) (Fig. 5a), it is possible to obtain spectrum $P_{N3}(\omega)$ of periodic signal $P_{N3}(t)$ (Fig. 5d)

All three spectrums $P_{N1}(\omega)$, $P_{N2}(\omega)$ and $P_{N3}(\omega)$ (Fig. 5b, Fig. 5c and Fig. 5d correspondingly) consist of different frequency components. Spectrum $P_{N1}(\omega)$ consists of one frequency component with $k = 0$, $P_{N2}(\omega)$ consists of three components, namely frequency components with $k = 3$, $k = 6$, and $k = 9$ and spectrum $P_{N3}(\omega)$ consists of eight frequency components with k = 1, 2, 4, 5, 7, 8, 10, and 11. Because signals $P_{N1}(t)$, $P_{N2}(t)$, and $P_{N3}(t)$ all use non-overlapping sets of values k, they are orthogonal signals with zero cross correlation.

In the frequency domain, the presentation of signals $S_{NT}(t)$ as the sum of periodic orthogonal pulse signals $P_{Ni}(t)$ corresponds to the separation frequency components $S_{NT}(\omega)$ (Fig. 3a, Fig. 4a, and Fig. 5a)) of signals $S_{NT}(t)$ (N=12) (Fig. 2a). The periodic orthogonal pulse signals $P_{Ni}(t)$ (3) consist of different groups of frequency components $\omega_k = k \cdot 2\pi/NT$, k = 0,1,2,3, …. There are no mutual frequency components in the different groups, which is why the periodic pulse signals $P_{Ni}(t)$ (3) are orthogonal signals with zero cross correlation. The number of different groups of frequency components is defined by the number of periodic pulse signals $P_{Ni}(t)$ (3) which depends on the number of multipliers n in equation (2). Different combinations of multipliers, for the same value of N, lead to the formation of different combinations of periodic pulse signals $P_{Ni}(t)$ (3) consisting of different groups of frequency components.

As an example, for N = 12 there are eight different multiplier combinations (8). Previously we presented samples for only three combination $N = 1 \cdot 12 = 12$, $N = 1 \cdot 3 \cdot 4 = 12$, and $N = 1 \cdot 4 \cdot 3$

= 12 (9), (10) and (11). Fig. 3, Fig. 4, and Fig. 5 show that all of these three combinations of multipliers correspond to three different combinations of periodic pulse signals $P_{Ni}(t)$ with three different groups of frequency components. The same results with completely different groups of frequency components can be obtained for other combinations of multipliers for N=12 (8).

Formation of Periodic Time Spread Pulse Signals with Perfect Autocorrelation

In reference[1], it was shown that after presenting PSPS as a linear composition of periodic pulse signals $P_{Ni}(t)$ (3) with zero cross correlation, it is possible to form (PTSPS) $S^{(s)}_N(t)$ with perfect autocorrelation. Such formation can be accomplished by either of the following methods:

a) Cyclic shifts Δ between periodic pulse signals P_{Ni} (where Δ is equivalent to the time shift on time interval T between $P_{Ni}(t)$ signals)
b) Changing the polarity (positive or negative) of the P_{Ni} signal
c) Combining both methods, i.e. by shifts Δ and changing polarity

In the case when **N = 1 · 12 = 12** (9), one can observe that it is possible to form PTSPS $S^{(s)}_N$ from PSPS S_N (7) by only changing the polarity of the one $P_{Ni}(t)$ signal. The following is an example for signal $P_{N1}(t)$,

$$\begin{array}{ll} i=1 & -P_{N1} = (-p_1,-p_1, \ldots\ldots,-p_1,-p_1) = (-1,-1, \ldots\ldots,-1,-1) \\ i=2 & P_{N2} = (p_2,-p_1, \ldots\ldots,-p_1,-p_1) = (11,-1, \ldots\ldots,-1,-1) \\ & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ & S^{(s)}_N = \quad -P_{N1} + P_{N2} \quad = (10,-2,-2, \ldots\ldots,-2,-2), \end{array} \qquad (13)$$

Where $S^{(s)}_N = -P_{N1} + P_{N2}$ is the PTSPS. In reference[2], it was shown that signals $S^{(s)}_N = ((N-2), -2, -2, \ldots, -2)$ are PTSPSs with perfect autocorrelation (Fig. 2d), where N (N > 1) are positive integers including prime numbers N_p. Notice, that the main difference between signals $S_{NT}(t)$ (PSPS) and $S^{(s)}_{NT}(t)$ (PTSPS) (Fig. 2a and Fig. 2d respectively) in the case when N = 1 · 12 = 12 is that signal $S_{NT}(t)$ (PSPS) has a positive frequency spectrum $P_{N1}(\omega)$, but signal $S^{(s)}_{NT}(t)$ (PTSPS) has a negative frequency spectrum $P_{N1}(\omega)$. This result means that PSPS and PTSPS have a different sign of their DC components on frequency $\omega = \omega_k = \omega_0 = 0$ (k=0) (Fig. 3a and Fig. 3d respectively). All of the other frequency components are the same for both of these signals.

For **N = 1 · 3 · 4 = 12** (10) and for **N = 1 · 4 · 3 = 12** (11), to form PTSPS, the polarity of signals $P_{Ni}(t)$ can be changed and/or the cyclic shift Δ = 1 for P_{Ni} can be used.

Case: N = 1 · 3 · 4 = 12

$$
\begin{aligned}
&i=1 \quad -P_{N1} &&= (-p_1,-p_1, \ldots\ldots, -p_1,-p_1) = (-1,-1,-1,-1,-1,-1,-1,-1,-1,-1,-1,-1)\\
&i=2 \quad P_{N2}(\Delta=1) &&= (-p_1, p_2,-p_1, \ldots, -p_1,-p_1) = (-1, 2,-1, -1, 2,-1,-1, 2,-1,-1, 2,-1)\\
&i=2 \quad P_{N3} &&= (p_3, 0, \ldots\ldots, 0) = (9, 0, 0, -3, 0, 0, -3, 0, 0, -3, 0, 0)\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&S^{(s)}_N &&= -P_{N1} + P_{N2}(\Delta=1) + P_{N3} = (7, 1, -2, -5, 1, -2, -5, 1, -2, -5, 1, -2)
\end{aligned} \tag{14}
$$

Case: N = 1 · 4 · 3 = 12

$$
\begin{aligned}
&i=1 \quad -P_{N1} &&= (-p_1,-p_1, \ldots\ldots, -p_1,-p_1) = (-1,-1,-1,-1,-1,-1,-1,-1,-1,-1,-1,-1)\\
&i=2 \quad P_{N2}(\Delta=1) &&= (-p_1, p_2,-p_1, \ldots, -p_1,-p_1) = (-1, 3,-1,-1,-1, 3,-1,-1,-1, 3,-1,-1)\\
&i=2 \quad P_{N3} &&= (p_3, 0, \ldots\ldots, 0) = (8, 0, 0, 0, -4, 0, 0, 0, -4, 0, 0, 0)\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&S^{(s)}_N &&= -P_{N1} + P_{N2}(\Delta=1) + P_{N3} = (6, 2,-2,-2, -6, 2,-2,-2,-6, 2,-2,-2)
\end{aligned} \tag{15}
$$

In (14) and (15), observe that signal $-P_{N1}$ is negative signal P_{N1}, signal P_{N2} ($\Delta = 1$) is signal P_{N2} (t) shifted on time interval t = T relative to signals P_{N1} (t) and P_{N3} (t), and the sum of the three signals $-P_{N1}$, P_{N2} ($\Delta = 1$), and P_{N3} is the PTSPS $S^{(s)}_N$ with perfect autocorrelation on period of repetition NT = 12.

Observe that PTSPS signals $S^{(s)}_N$ (14) and (15) are different than PTSPS signal $S^{(s)}_N$ for N = 12 ($N = 3 \cdot 4 = 12$ and for $N = 4 \cdot 3 = 12$) obtained in.[1] By using signal P_{N1} with sign of plus in (14) and (15), i.e. signal P_{N1} instead of negative signal $-P_{N1}$, it is possible to obtain PTSPS $S^{(s)}_N$ for $N = 1 \cdot 3 \cdot 4 = 12$ and for $N = 1 \cdot 4 \cdot 3 = 12$ in a manner similar to the method used in.[1]

The introduction of multiplier 1 in (2) is a more general approach compared with the use of multiplication (2) without multiplier 1.[1] Because, in this case, it is possible to use presentation of PSPS (1) as a linear composition of periodic orthogonal pulse signals (3) for any integers N (N > 1) including prime numbers N_p. Additionally, the introduction of multiplier 1 (2) has a clear frequency interpretation because the multiplier $N_1 = 1$ (2) corresponds to DC signals, namely signals P_{N1} (t) (Fig. 2b) (3) with zero frequency P_{N1} (ω) (Fig. 3b, Fig. 4b and Fig. 5b).

As shown above, after presentation of PSPS S_{NT} (t) as the sum of periodic signals with zero cross correlation P_{Ni} (t) (9), (10) and (11), it is possible to form the PTSPS with perfect autocorrelation (13), (14) and (15). This formation may be accomplished by changing the polarity of periodic signals P_{Ni} (t) (9), (13) and/or changing the time shifts between periodic signals P_{Ni} (t) (10), (13) and (11), (14). Both formation methods detailed above, i.e. changing polarity and changing time shifts, are related to changing phases of frequency components which belong to different periodic signals P_{Ni} (t).

The following steps detail the procedure used to form PTSPS from PSPS (1).

1. Create a presentation of PSPS in the form of (1). The duration of rectangular single pulse T and the period of repetition T_{rep} = NT are the characteristics of electronic system.
2. Create a presentation N as a multiplication (product) of several multipliers (factors) N_i (2), (9), (10) and (11).
3. Create a presentation PSPS (1) as a linear composition of periodic signals with zero cross correlation P_{Ni} (t) (3), (9), (10) and (11).
4. Exploit changing the polarity of periodic signals P_{Ni} (t) and/or time shifts between periodic signals P_{Ni} (t) (13), (14) and (15).
5. Summarize all of the periodic signals P_{Ni} (t) (13), (14) and (15) in order to obtain PTSPS.

Notice that multiplications (2) consisted of three or more multipliers correspond to different combinations of multipliers N_i, and they subsequently correspond to different combinations of signals P_{Ni} (t). The time shifts (phase changes) between P_{Ni} (t) also vary by different amounts. This variety of combinations leads to a variety of PTSPS for the same value of N [1]. The final choice of PTSPS from this variety depends on real-life conditions, i.e. it depends on the working conditions of transmitters, receivers, and possible distortions of transmitted signals.

It is important to emphasize that PSPS S_{NT} (t) and PTSPS $S^{(s)}_{NT}$ (t) (Fig. 1a and Fig. 1c) have the same frequency spectrum width. The spectrum width is defined by the duration of rectangle single pulse T and equals 2π/T (Fig. 3, Fig. 4, and Fig. 5). The spectrums of both signals (PSPS) S_{NT} (t) and (PTSPS) $S^{(s)}_{NT}$ (t) only differ in the phases of their frequency components.

Properties of Periodic Signals P_{Ni} (t)

Periodic pulse signals P_{Ni} (t) (3) have interesting properties related with the presentation PSPS as a linear composition of periodic pulse signals P_{Ni} (t), and these properties are not related with the formation of PTSPS from PSPS.

The first interesting property is related with the zero cross correlation of periodic pulse signals P_{Ni} (t). Because periodic pulse signals P_{Ni} (t) have no mutual frequencies, these types of signals can be used for transmitting information from sources to recipients independently from each other, and is similar to using different frequency channels to transmit information. On the receiver side, the periodic pulse signals P_{Ni} (t) can easily be separated. Notice that for

$N = 2^n$, it is possible to show that periodic pulse signals $P_{Ni}(t)$ can be presented as orthogonal binary Walsh signals with zero cross correlation.

The second interesting property is related with the autocorrelation functions of periodic pulse signals $P_{Ni}(t)$. Reference[1] showed that periodic pulse signals $P_{Ni}(t)$ (3) have autocorrelation functions with one positively-signed main lobe and several negatively-signed side lobes. These types of signals can be used for time synchronization purposes in communication, navigation and telemetric systems. There is no negative influence by negative side lobes on the synchronization procedure in this case. These synchronization channels will be independent time synchronization channels, because periodic pulse signals $P_{Ni}(t)$ (3) are orthogonal signals with zero cross correlation.

Based on the above properties, observe that periodic pulse signals $P_{Ni}(t)$ (3) can be used in MIMO (multiple input and multiple output) systems. In MIMO systems, information sent from many sources to many recipients can be transmitted simultaneously through the same frequency channel, and can easily be separated by recipients.

Finally, in wireless systems (coherent radio signals systems), direct current (DC) signals $P_{N1}(t)$ (Fig. 2b) are very stable carrier frequency signals. In[1], it was stated that these carrier frequency signals can be used, as a part of PTSPS $S^{(s)}_{NT}(t)$, to measure Doppler speed with an accuracy of 1/NT in radar applications. These stable carrier frequency signals, as part of PTSPS $S^{(s)}_{NT}(t)$, also can be used to tune the frequency and phase of a receiver to the frequency and phase of incoming signals in radio navigation, wireless communication and telemetric systems.

CONCLUSION

It has been shown that periodic single pulse signals $S_{NT}(t)$ (PSPS) of duration T with a period of repetition $T_{rep} = NT$ $(N > 1)$, where N are positive integers (including prime numbers $N = N_p$), can be presented as a linear composition of periodic orthogonal pulse signals with zero cross correlation. The frequency interpretation of this presentation was performed.

It has also been shown that after PSPS has been presented as a linear composition, these signals can be transformed into periodic time spread pulse signals $S^{(S)}_{NT}(t)$ (PTSPS), where the energy of the original periodic single pulse signals (PSPS), which was concentrated on a narrow interval T, is spread over the entire interval NT. During this transformation procedure, the primary property of PSPS, namely periodic perfect ACF on interval NT, does not change. The properties of periodic orthogonal pulse signals with zero cross correlation are discussed.

The results of this study advance the theory of pulse signals and can be applied to any periodic pulse signals systems including radar, radio navigation, communication and telemetric systems.

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