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# REMARKS ON MULTIPLICATIVE GENERALIZED DERIVATIONS IN PRIME AND SEMIPRIME RINGS

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ABSTRACT

Let $\boldsymbol{R}$ be a semiprime ring with center $\boldsymbol{Z}(\boldsymbol{R})$ and $\boldsymbol{L}$ a nonzero left ideal of $\boldsymbol{R}$. A map $\boldsymbol{F}:\boldsymbol{R}\rightarrow\boldsymbol{R}$ (not necessarily additive) is called multiplicative generalized derivation if it satisfies $\boldsymbol{F}(\boldsymbol{xy})=\boldsymbol{F}(\boldsymbol{x})\boldsymbol{y}+\boldsymbol{x}\boldsymbol{g}(\boldsymbol{y})$ for all $\boldsymbol{x},\boldsymbol{y}\in\boldsymbol{R}$, where $\boldsymbol{g}:\boldsymbol{R}\rightarrow\boldsymbol{R}$ a derivation. The main aim in this paper is to study the following situations: $(P_1)\ F(xy)-F(x)g(y)\in Z(R)$ and $(P_2)\ F(xy)+g(y)F(x)\in Z(R)$ for all $x,y\in R$ in some appropriate subset of $R$.

**KEYWORDS:** Left ideal, multiplicative generalized derivation, semiprime ring.
MSC 2010, Classifications: 16N60, 16W25, 16Y30.

## INTRODUCTION

Throughout the present paper; $R$ will denote an associative ring with centre $Z(R)$.

For given $x,y\in R$, the symbol $[x,y]$ and $xoy$ denote the commutator $xy-yx$ and anticommutator $xy+yx$, respectively. Posner[12] enumerated two outstanding results on derivation in prime ring, these results stated that: (i) In a 2-torsion-free prime ring, if the iterate of two derivations is a derivation, then one of them must be zero; (ii). A prime ring R admitting a nonzero centralizing derivation d must be commutative. Since then, derivation in ring have been generalized in different direction such as Jordan derivation, left derivation, (θ,φ)-derivation, generalized derivation, generalized Jordan derivation, generalized Jordan (θ,φ)-derivation, higher derivations, generalized higher derivations and others.

Moreover, generalized derivations got it motivation from Bresar,[2] who acknowledged the distance of composition of two derivations. Bresar,[2] estimated the distance of the composition of two derivations to the generalized derivations. Over last few decades a lot of works has been done on generalized derivation (see for references [1, 3, 5, 6]). The concept of generalized derivation covers both concept of derivation and that of a left multiplier that is an additive mapping $f\colon R \to R$ satisfying $f(xy) = f(x)y$ for all $x, y \in R$.

Throughout this paper, a ring R represents an associative ring (not necessarily unity) with Z(R), centre of R. In this sequel, we need the basic definitions as given below.

- An additive mapping $d : R \to R$ is a derivation if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$.
- An additive mapping $F : R \to R$ is called a generalized derivation of R, if there exist a derivation g: R→R such that $F(xy) = F(x)y + xg(y)$ holds for all $x, y \in R$.

**Remarks 1.1.** Hvala,[8] initiated generalized derivations from the algebraic viewpoint and defined that an additive map $f$ of a ring R into itself will be called a generalized derivation if there exist a derivation d of R such $f(xy) = f(x)y + xd(y)$ for all $x, y \in R$.
Next, Lee and Shieu[10] extended the definition of generalized derivations as follows:

By a generalized derivation we mean an additive mapping $g: I \to U$ such that $g(xy)=g(x)y+xd(y)$ for all $x, y \in I$, where I is a dense right ideal of R and d is a derivation from I into U.

- A mapping $D: R \to R$ (not necessarily additive) which satisfies $D(xy) = D(x)y + xD(y)$ *for all* $x, y \in R$ is called multiplicative derivation of R.
- A mapping $F : R \to R$ is called a multiplicative (generalized)- derivation if $F(xy) = F(x)y + xg(y)$ is fulfilled for all $x, y \in R$, where g: R→R is a derivation.

Multiplicative generalized derivations precisely have been studied by different authors in different directions in the past few years (see for references.[5,6,8]). In this line of investigation, Martindale[11] posed a well-known problem: "When are multiplicative mappings additive?" This question was motivated by the work of Rickart[13] and Johnson.[9]

Furthermore, Rickart.[13] raised the question of "when a multiplicative isomorphism additive"? In both of these papers, Martindale.[4] noted some kind of minimality conditions imposed on the ring R, generalized the hypothesis of Rickart at the same time omit the minimality conditions and surveyed this result of Rickart in the presence of family of idempotent elements as: Let $S$ be a nonempty subset of a ring $R$.The mapping $f : S \rightarrow R$ is a centralizing (or commuting) map on $S$ if $[f(x), x] \in Z(R)$ (or $[f(x), x] = 0$) $\forall\, x \in S$. Martindale,[11] answered the question of when a multiplicative mapping additive and stated that: Suppose R is a ring containing a family $\{e_a \in A\}$ of idempotent that satisfies:

I) $xR = 0$ implies $x = 0$;

II) $e_a \mathrm{R}\, x = 0\ for\ each\ a \in A, then\ \ x = 0$(and hence $Rx = 0$ implies $\mathrm{x} = 0$);

III) For each $a \in A$, $e_a x \mathrm{e_a} R(1 - e_a) = 0$ implies $e_a x e_a = 0$.

Then any multiplicative isomorphism ∅ of R onto an arbitrary ring S is additive.

In his paper,[4] Daif introduced the idea of multiplicative derivation and gave the precise definition of the term 'multiplicative derivation'.Daif and Tamman El-Sayyiad[3] generalized the definition of multiplicative derivation to multiplicative generalized derivations.

A natural question of "when a multiplicative derivation additive" was answered by Daif,[4] was motivated by the work of Martindale[11] and introduced the notion of multiplicative derivation as: The mapping $D: R \rightarrow R$ is said to be a multiplicative derivation if it satisfies D(xy)=D(x)y +x**D(y)** for all x, y ∈ R. in the case of multiplicative derivations, the mappings assumed not to be an additive mapping. Further, Goldmann and Semrl[7] gave the complete description of these mappings.

Daif and Tammam El-sayiad[3] extend multiplicative derivations to multiplicative generalized derivations as follows: a mapping F on R is said to be a multiplicative (generalized)-derivation if there exists a derivations d on R such that F(xy) = F(x)y + xd(y) for all $x, y \in R$.

The concepts of multiplicative generalized derivation cover the concepts of multiplicative derivation and multiplicative generalized derivation.

In this definition, if we take d to be a mapping that is not necessarily additive and not necessarily derivation then F is called a multiplicative (generalized)- derivation which was introduced by Dhara and Ali.[6]

Recently, Dhara and Ali.[6] gave a precise definition of multiplicative generalized derivation as follows: a mapping $F : R \to R$ is said to be a multiplicative (generalized)- derivation if there exist a map g on R such that $F(xy) = F(x) + xg(y)$ for all $x, y \in R$ where g is any mapping on R (not necessarily additive).

A multiplicative (generalized)- derivation associated with mapping g = 0 covers the concepts of multiplicative centralizers (not necessarily additive).

The example of multiplicative generalized derivations are multiplicative derivation and multiplicative centralizers (see Dhara and Ali.[7] for further reference).

On the other hand, Daif studied this situation of Martindale and proved a similar result by replacing the mapping $\emptyset$ with multiplicative derivations.

In 2018, Dhara and Mozumder[5] investigated the commutativity of semiprime ring admitting a multiplicative (generalized)- derivation satisfying the following differential properties:

(i) $F(xy) + F(x)F(y) \in Z(R)$

(ii) $F(xy) - F(x)F(y) \in Z(R)$

(iii) $F(xy) + F(y)F(x) \in Z(R)$

(iv) $F(xy) - F(y)F(x) \in Z(R)$

(v) $F(xy) - g(y)F(x) \in Z(R)$. for all $x,y \in Z(R)$

In this line of investigation, it is natural to ask the question of related identities (i) - (v) to establish commutativity of prime and semi prime rings involving multiplicative generalized derivations. Motivated by these works, the aim of this article is to establish commutativity of semiprime rings admitting a multiplicative (generalized)- derivation associated with a nonzero derivation $g$ satisfying the following identities $(P_1)$ and $(P_2)$ for all $x, y$ in some suitable subset of $R$.

## 1. Preliminaries results

We shall make use of the following basic identities frequently in the prove of our results

(i) $[xy, z] = x[y, z] + [x, z]y$

(ii) $[x, yz] = y[x,z] + [x,y]z$ for all $x, y, z \in R$.

In order to prove our results, we need the following results.

**Lemma 1.1**. ([5, Lemma 2])
(a) If R is a semiprime ring, the centre of a nonzero one-sided ideal is contained in the centre of R; in particular, any commutative one-sided ideal is contained in the centre of R.
(b) If R is a prime ring with a nonzero central ideal, then R must be commutative.

**Lemma 1.2**. ([4, Theorem 3]) Let R be a semiprime ring and U a nonzero left ideal of R. If R admits a derivation d which is nonzero on U and centralizing on U, then R contains a nonzero central ideal.

**Lemma 1.3**. ([6, Lemma 2]) If R is prime with a nonzero ideal, then R is commutative.

**Lemma 1.4**. ([3, Theorem 4]) Let R be a central prime ring and I be a nonzero left ideal of R. If R admits a nonzero derivation d which is centralizing on I, then R is commutative.

**Lemma 1.5**. ([10, Theorem 2 (ii)]) Let R be a noncommutative prime ring with extended centroid C, λ a nonzero left ideal of R and p, q, r, k are fixed positive integers. If d is a derivation of R such that $x^p[d(x^q), x^r]^k = 0$ for all $x \in \lambda$, then either $d = ad(b)$ and $\lambda b = (0)$ for some $b \in Q$ or $\lambda[\lambda, \lambda] = (0)$ and $d(\lambda) \subseteq \lambda C$.

**Lemma 1.6**. ([4, Fact-4]) Let R be a semiprime ring, d a nonzero derivation of R such that $x[[d(x), x], x] = 0$ for all $x \in R$. Then d maps R into its centre.

**Lemma 1.7**. ([5, Lemma 2.4]) If R is a prime ring, d: R → R a derivation of R, I a nonzero left ideal of R and $0 \neq a \in R$ such that $[ad(x), x] = 0$ for all $x \in I$, then one of the following holds: (1) $a \in Z(R)$; (2) $Ia = (0)$; (3) $Id(I) = (0)$.

## 2. Main results

We are now in a position to prove our main theorems.

**Theorem 2.1** Let R be a semi prime ring with centre $Z(R)$ and $A$ a nonzero left ideal of R. let $F: R \to R$ be a multiplicative (generalized)- derivation associated with the derivation $\delta : R \to R$. if $F(xy) - F(x)\delta(y) \in Z(R)$ *for all* $x, y \in A$, *then* $A[\delta(x), x] = (0)$. For all $x \in A$. In particular, when $A = R$, then either $\delta = 0$ or R contains a nonzero central ideal.

**Proof.** By hypothesis, we have

$F(xy) - F(x)\delta(y) \in Z(R).\ \forall\, x, y\ \in A$ (2.1)

Replacing y with $yz$ in (2.1), we obtain.

$F(xyz) - F(x)\delta(yz)\ \in Z(R).\ y, z\ \in A.$ (2.2)

Or,

$F(xyz) - F(x)[\delta(y)z + y\delta(z)]\ \in Z(R)$ (2.3)

This gives

$F(xyz) - F(x)\delta(y)z - F(x)y\delta(z)\ \in Z(R).$ (2.4)

This can be re-written as

$F(xyz) - F(xy)z + [F(xy) - F(x)\delta(y)]z - F(x)y\delta(z) \in Z(R)$ (2.5)

$[F(xyz), z] - [F(xy), z]z + [F(xy) - F(x)\delta(y), z]z - F(x)y\delta(z) = 0.$ (2.6)

Since $F(xy) - F(x)\delta(y)\ \in Z(R)$ , then the above equation reduces to

$[F(xyz), z] - [F(xy), z]z - F(x)y\delta(z) = 0.\ \forall\, x, y\ \in A$ (2.7)

Replacing $x = xz\ in\ (2.7)$, we have

$[F(xzyz), z] - [F(xzy), z]z - [F(xz)y\delta(z), z] = 0..$ (2.8)

$[F(xzyz), z] - [F(xzy), z]z - [F(x)z + x\delta(z))y\delta(z), z] = 0.$ (2.9)

Putting $y = zy\ in\ (2.9)$ we have

$[F(xzyz), z] - [F(xzy), z]z - [F(x)zy\delta(z), z] = 0.$ (2.10)

Subtracting (2.9) from (2.10) we have

$[F(xzyz), z] - [F(xzy), z]z - [F(x)z + x\delta(z))y\delta(z), z] - [F(xzyz), z] - [F(xzy), z]z -$
$[F(x)zy\delta(z), z] = 0$ (2.11)

This reduces to

$[x\delta(z)y\delta(z), z] = 0..\ \forall\, x, y\ \in A$ (2.12)

Putting $x = \delta(z)x$ in (2.12) we have

$[\delta(z)x\delta(z)y\delta(z), z] = 0$ (2.13)

$\delta(z)[x\delta(z)y\delta(z), z] + [\delta(z), z]x\delta(z)y\delta(z)$ (2.14)

By application (2.12), (2.14) reduces to

$[\delta(z), z]x\delta(z)y\delta(z) = 0.$ (2.15)

Right multiplication of (2.15) by u for some $u\ \in A$ we have

$u[\delta(z), z]x\delta(z)y\delta(z) = 0$ (2.16)

Replacing $y = yt$ for some $t\ \in A$, we obtain

$u[\delta(z), z]x\delta(z)yt\delta(z) = 0$ (2.17)

Replacing $t = tz$ *in* (2.17), gives

$u[\delta(z), z]x\delta(z)ytz\delta(z) = 0$ (2.18)

Right multiplication of (2.17) by z we have

$u[\delta(z), z]x\delta(z)yt\delta(z)z = 0$ (2.19)

Subtracting (2.18) from (2.19) we have

$u[\delta(z), z]x\delta(z)yt\delta(z)z - u[\delta(z), z]x\delta(z)ytz\delta(z) = 0.$

This yields

$u[\delta(z), z]x\delta(z)yt[\delta(z), z] = 0.$ (2.20)

Replacing $x = xz$ in(2.20), we get

$u[\delta(z), z]xz\delta(z)yt[\delta(z), z] = 0$ (2.21)

Replacing $y = zy$ in (2.21), we obtain

$u[\delta(z), z]x\delta(z)zyt[\delta(z), z] = 0.$ (2.22)

Subtracting (2.21) from (2.22), we get

$u[\delta(z), z]x[\delta(z), z]yt[\delta(z), z] = 0$ (2.23)

Putting $v = yt$, we get

$u[\delta(z), z]x[\delta(z), z]v[\delta(z), z] = 0. \ \forall u, x, z, v \in A$ (2.24)

$\Rightarrow A[\delta(z), z]A[\delta(z), z]A[\delta(z), z] = 0$, this implies $0 = (A[\delta(z), z])^3$

since R is semiprime, it contains no nonzero nilpotent right ideal, implying $(A[\delta(z), z])=0$.

For all z ∈ A. In particular when $A = R$, then $[\delta(x), x] = 0 \ \forall x \in R$.

**Theorem 2.2**: Let R be a semi prime ring with centre $Z(R)$ and A a nonzero left ideal of R. let $F: R \to R$ be a multiplicative (generalized)- derivation associated with derivation $\delta: R \to R$. If

F(xy)+**δ(y) F (x)** εZ (R) $\forall x, y \in A$, then $A[\delta(x), x] = (0)$.

For all $x \in A$, in particular, when $A = R$, then either $\delta = 0$ or R contains a nonzero central ideal.

**Proof.**

Given F(xy)+**δ(y) F (x)** εZ (R). $\forall x, y \in A$ (2.25)

Replacing $x$ with $xz$ where $z \in A$, we have

$F(xzy) + \delta(y)F(xz) \in Z(R) \forall x, y, z \in A$. (2.26)

This implies that

$F(xzy)+\delta(y)[F(x)z+x\delta(z)]\ \in\ Z(R)$. (2.27)

Or

$F(xzy)+\delta(y)F(x)z+\delta(y)x\delta(z)\ \in Z(R). \forall x,y,z \in A$ (2.28)

One can write (2.28), as

$F(xzy)-F(xy)z+[F(xy)+\delta(y)F(x)]z+\delta(y)x\delta(z)\ \in Z(R)$ (2.29)

Commuting both sides with z we have

$[\,F(xzy)-F(xy)z+[F(xy)\ +\delta(y)F(x)]z+\delta(y)xg),z]=0$ (2.30)

By the given condition, (2.30), becomes

$[\,F(xzy)-F(xy)z+\delta(y)x\delta(z),z]=0.\forall\ x,y,z \in A$ (2.31)

Replacing $y\ by\ z$, we get

$[F(xz^2)-F(xz)z+\delta(z)x\delta(z),z]=0.\forall\ x,z \in A$ (2.32)

Or

$[F(x)z^2+x\delta(z^2)-[(F(x)z+x\delta(z)]z+\delta(z)x\delta(z)),z]=\ 0$ (2.33)

We have that

$[xz\delta(z)+\delta(z)x\delta(z),z]=0$ (3.34)

Replacing $x$ by $zx$ in (4.34), we have

$[zxz\delta(z)+\delta(z)zx\delta(z),z]=0.$ (3.35)

Left multiplication of (4.34) by z and then, subtracting from (4.35), we obtain

$[[\delta(z),z]x\delta(z),z]=0$ (3.36)

By application of (Dhara and Mozumder [5], Theorem 3.6), we find that

$[\delta(z),z]^2\ x[\delta(z),z]^2u[\delta(z),z]^2\ =\ 0\ for\ all\ x,u,z\ \in A.$

This implies that

$(A[\delta(z),z]^2)^3\ =\ (0)\ for\ all\ z\ \in\ A.$

Since R is semiprime, we conclude that

A $[d(z), z]^2$= (0) for all z ∈ $A$.

In particular, when $A$ = R, then

A $[d(z), z]^2$= (0) for all $x\ \in\ R$, either δ =0 or R contains a nonzero central ideal.

## 3. Counter example

In this section, we construct an example of a derivation and multiplicative generalized derivation that illustrate our theorems.

Let R = $\left\{ \begin{pmatrix} 0 & 0 & 0 \\ \alpha & 0 & 0 \\ \beta & \gamma & 0 \end{pmatrix} \middle| \alpha, \beta, \gamma \in \mathbb{Z} \right\}$, where $\mathbb{Z}$ is the set of all integers.

Since $\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} R \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} = (0)$, so $R$ is not prime ring. Define the mapping $d$ and $F: R \to R$

As follows: $d\begin{pmatrix} 0 & 0 & 0 \\ \alpha & 0 & 0 \\ \beta & \gamma & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \beta & \gamma & 0 \end{pmatrix}$ and $F\begin{pmatrix} 0 & 0 & 0 \\ \alpha & 0 & 0 \\ \beta & \gamma & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ -\alpha & 0 & 0 \\ \beta & \gamma & 0 \end{pmatrix}$. It suffices to verify that

d and F are derivation and multiplicative generalized derivation in $R$ such that

$F(xy) = F(x)y + xd(y)$ Holds for all $x, y \in R$.

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