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ON ZEROS OF POLYNOMIALS

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ABSTRACT

In this paper we find ring-shaped regions containing all or a specific number of zeros of a polynomial. Many important results follow easily from our results.

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KEYWORDS AND PHRASES: Polynomial, Region, Zeros.

INTRODUCTION

A classical result on the location of zeros of a polynomial is the following known as the Enestrom-Kakeya Theorem:[2,3]

Theorem A: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n such that

$$a_n \geq a_{n-1} \geq ...... \geq a_1 \geq a_0 > 0.$$

Then all the zeros of P(z) lie in $|z| \leq 1$.

Another classical result giving a region containing all the zeros of a polynomial is the following known as Cauchy's Theorem:[2,3]

Theorem B: All the zeros of the polynomial $P(z)=\sum_{j=0}^{n} a_j z^j$ of degree n lie in the circle

$$|z| \leq 1+M, \text{ where } M = \max_{0\leq j\leq n-1}\left|\frac{a_j}{a_n}\right|.$$

The above theorems have been generalized in various ways by the researchers.

MAIN RESULTS

In this paper we prove the following:

Theorem 1: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n and

$$L=|a_n-a_{n-1}|+|a_{n-1}-a_{n-2}|+......+|a_1-a_0|+|a_0|.$$

Then all the zeros of P(z) lie in $\frac{|a_0|}{R^{n+1}[|a_n|+L-|a_0|]}\le|z|\le\frac{L}{|a_n|}$ for $R\ge1$

and in $\frac{|a_0|}{R[|a_n|+L-|a_0|]}\le|z|\le\frac{L}{|a_n|}$ for $R\le1$, provided $|a_n|\le L$.

Further the number of zeros of P(z) in $\frac{|a_0|}{R^{n+1}[|a_n|+L-|a_0|]}\le|z|\le\frac{R}{c}, c>1$ does not exceed $\frac{1}{\log c}\log\frac{|a_0|+R^{n+1}[|a_n|+L-|a_0|]}{|a_0|}$ for $R\ge1$ and the number of zeros of P(z) in $\frac{|a_0|}{R[|a_n|+L-|a_0|]}\le|z|\le\frac{R}{c}, c>1$ does not exceed $\frac{1}{\log c}\log\frac{|a_0|+R[|a_n|+L-|a_0|]}{|a_0|}$ for $R\le1$.

Remark 1: If $a_n\ge a_{n-1}\ge......\ge a_1\ge a_0>0$, then $L=|a_n|$ and it follows from Theorem 1 that all the zeros of P(z) lie in $|z|\le1$, which is Theorem A i.e. the Enestrom-Kakeya Theorem.

If we take R=1 in Theorem 1, we get the following result:

Corollary 1: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n and

$$L=|a_n-a_{n-1}|+|a_{n-1}-a_{n-2}|+......+|a_1-a_0|+|a_0|.$$

Then the number of zeros of P(z) in $\frac{|a_0|}{|a_n|+L-|a_0|}\le|z|\le\frac{R}{c}, c>1$ does not exceed

$$\frac{1}{\log c}\log\frac{(L+|a_n|)}{|a_0|}.$$

Instead of proving Theorem 1, we prove the following more general result:

Theorem 2: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with $\mathrm{Re}(a_j)=\alpha_j, \mathrm{Im}(a_j)=\beta_j, j=0,1,2,......,n$ and

$$L = |\alpha_n - \alpha_{n-1}| + |\alpha_{n-1} - \alpha_{n-2}| + ... + |\alpha_1 - \alpha_0| + |\alpha_0|$$
$$M = |\beta_n - \beta_{n-1}| + |\beta_{n-1} - \beta_{n-2}| + ... + |\beta_1 - \beta_0| + |\beta_0|.$$

Then all the zeros of P(z) lie in $\frac{|a_0|}{R^{n+1}[|a_n| + L + M - |\alpha_0| - |\beta_0|]} \le |z| \le \frac{L+M}{|a_n|}$ for $R \ge 1$

and in $\frac{|a_0|}{R[|a_n| + L + M - |\alpha_0| - |\beta_0|]} \le |z| \le \frac{L+M}{|a_n|}$ for $R \le 1$, provided $|a_n| \le L + M$.

Further the number of zeros of P(z) in $\frac{|a_0|}{R^{n+1}[|a_n| + L + M - |\alpha_0| - |\beta_0|]} \le |z| \le \frac{R}{c}, c > 1,$ does not exceed

$\frac{1}{\log c} \log \frac{|a_0| + R^{n+1}[|a_n| + L + M - |\alpha_0| - |\beta_0|]}{|a_0|}$ for $R \ge 1$ and the number of zeros of P(z) in

$\frac{|a_0|}{R[|a_n| + L + M - |\alpha_0| - |\beta_0|]} \le |z| \le \frac{R}{c}, c > 1,$ does not exceed

$\frac{1}{\log c} \log \frac{|a_0| + R[|a_n| + L + M - |\alpha_0| - |\beta_0|]}{|a_0|}$ for $R \le 1$.

Remark 2: Taking a_j real i.e. $\beta_j = 0, \forall j = 0,1,2,.....,n$, Theorem 2 reduces to Theorem 1.

If we take R=1 in Theorem 2, we get the following result:

Corollary 2: Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with $\text{Re}(a_j) = \alpha_j, \text{Im}(a_j) = \beta_j, j = 0,1,2,......,n$ and

$$L = |\alpha_n - \alpha_{n-1}| + |\alpha_{n-1} - \alpha_{n-2}| + ... + |\alpha_1 - \alpha_0| + |\alpha_0|$$
$$M = |\beta_n - \beta_{n-1}| + |\beta_{n-1} - \beta_{n-2}| + ... + |\beta_1 - \beta_0| + |\beta_0|.$$

Then the number of zeros of P(z) in $\frac{|a_0|}{|a_n| + L + M - |\alpha_0| - |\beta_0|} \le |z| \le \frac{1}{c}, c > 1,$ does not exceed

$$\frac{1}{\log c} \log \frac{|a_0| + |a_n| + L + M - |\alpha_0| - |\beta_0|}{|a_0|}.$$

LEMMAS

For the proof of Theorem 2, we need the following results:

Lemma 1: Let f(z) (not identically zero) be analytic for $|z| \le R, f(0) \ne 0$ and $f(a_k) = 0,$ $k = 1,2,......,n$. Then

$$\frac{1}{2\pi}\int_0^{2\pi} \log\left|f(\mathrm{Re}^{i\theta}\right| d\theta - \log|f(0)| = \sum_{j=1}^{n} \log\frac{R}{|a_j|}.$$

Lemma 1 is the famous Jensen's Theorem (see page 208 of [1]).

Lemma 2: Let f (z) be analytic for $|z| \le R, f(0) \ne 0$ and $|f(z)| \le M$ for $|z| \le R$. Then the number of zeros of f(z) in $|z| \le \frac{R}{c}, c > 1$ does not exceed $\frac{1}{\log c}\log\frac{M}{|f(0)|}$.

Lemma 2 is a simple deduction from Lemma 2.

PROOFS OF THEOREMS

Proof of Theorem 2: Consider the polynomial

$$F(z) = (1-z)P(z)$$

$$= (1-z)(a_n z^n + a_{n-1}z^{n-1} + ... + a_1 z + a_0)$$

$$= -a_n z^{n+1} + (a_n - a_{n-1})z^n + ... + (a_{\lambda+1} - a_\lambda)z^{\lambda+1} + (a_\lambda - a_{\lambda-1})z^\lambda$$

$$+ ...... + (a_1 - a_0)z + a_0$$

$$= -a_n z^{n+1} + (\alpha_n - \alpha_{n-1})z^n + ... + (\alpha_1 - \alpha_0)z + \alpha_0 + i\{(\beta_n - \beta_{n-1})z^n + ...$$

$$+ (\beta_1 - \beta_0)z + \beta_0\}$$

For $|z| > 1$ so that $\frac{1}{|z|^j} < 1, \forall j = 1,2,......,n$, we have, by using the hypothesis

$$|F(z)| \ge |a_n||z|^{n+1} - \{|\alpha_n - \alpha_{n-1}||z|^n + ... + |\alpha_1 - \alpha_0||z| + |\alpha_0| + |\beta_n - \beta_{n-1}||z|^n + ...$$

$$+ |\beta_1 - \beta_0||z| + |\beta_0|\}$$

$$= |z|^n[|a_n||z| - \{|\alpha_n - \alpha_{n-1}| + \frac{|\alpha_{n-1} - \alpha_{n-2}|}{|z|}. + ... + \frac{|\alpha_1 - \alpha_0|}{|z|^{n-1}} + \frac{|\alpha_0|}{|z|^n} + |\beta_n - \beta_{n-1}|$$

$$+ \frac{|\beta_n - \beta_{n-1}|}{|z|} + ... + \frac{|\beta_1 - \beta_0|}{|z|^{n-1}} + \frac{|\beta_0|}{|z|^n}\}]$$

$$> |z|^n[|a_n||z| - \{|\alpha_n - \alpha_{n-1}| + |\alpha_{n-1} - \alpha_{n-2}|. + ... + |\alpha_1 - \alpha_0| + |\alpha_0| + |\beta_n - \beta_{n-1}|$$

$$+ |\beta_n - \beta_{n-1}| + ... + |\beta_1 - \beta_0| + |\beta_0|\}]$$

$$= |z|^n [|a_n||z| - (L+M)]$$
$$> 0$$

if

$$|z| > \frac{L+M}{|a_n|}$$

provided $|a_n| \le L+M$.

This shows that those zeros of F(z) whos modulus is greater than 1 lie in $|z| \le \frac{L+M}{|a_n|}$.

Since the zeros of F(z) whose modulus is less than or equal to 1 already satisfy the above inequality, it follows that all the zeros of F(z) and hence all the zeros of P(z) lie in $|z| \le \frac{L+M}{|a_n|}$.

On the other hand, we have

$$F(z) = -a_n z^{n+1} + a_0 + (\alpha_n - \alpha_{n-1})z^n + ... + (\alpha_1 - \alpha_0)z + i\{(\beta_n - \beta_{n-1})z^n + ...$$
$$+ (\beta_1 - \beta_0)z\}$$
$$= a_0 + G(z)$$

Where $G(z) = -a_n z^{n+1} + (\alpha_n - \alpha_{n-1})z^n + ... + (\alpha_1 - \alpha_0)z + i\{(\beta_n - \beta_{n-1})z^n + ...$

$$+ (\beta_1 - \beta_0)z\}.$$

For $|z| = R$, we have, by using the hypothesis

$$|G(z)| \le |a_n||z|^{n+1} + |\alpha_n - \alpha_{n-1}||z|^n + ... + |\alpha_1 - \alpha_0||z| + |\beta_n - \beta_{n-1}||z|^n + ... + |\beta_1 - \beta|_0 |z|$$
$$= |a_n| R^{n+1} + |\alpha_n - \alpha_{n-1}| R^n + ... + |\alpha_1 - \alpha_0| R + |\beta_n - \beta_{n-1}| R^n + ... + |\beta_1 - \beta|_0 R$$
$$\le R^{n+1}[|a_n| + |\alpha_n - \alpha_{n-1}| + ... + |\alpha_1 - \alpha_0| + |\beta_n - \beta_{n-1}| + ... + |\beta_1 - \beta_0|]$$
$$= R^{n+1}[|a_n| + L + M - |\alpha_0| - |\beta_0|]$$

for $R \ge 1$.

For $R \le 1$,

$$|G(z)| \le R[|a_n| + L + M - |\alpha_0| - |\beta_0|].$$

Since G(z) is analytic for $|z| \le R, G(0) = 0$, it follows by Schwarz Lemma that in $|z| \le R$,

$$|G(z)| \le R^{n+1}[|a_n| + L + M - |\alpha_0| - |\beta_0|]|z| \quad \text{for } R \ge 1 \text{ and}$$

$$|G(z)| \le R[|a_n| + L + M - |\alpha_0| - |\beta_0|]|z| \quad \text{for } R \le 1$$

Hence for $|z| \le R$,

$$|F(z)| = |a_0 + G(z)|$$

$$\ge |a_0| - |G(z)|$$

$$\ge |a_0| - R^{n+1}[|a_n| + L + M - |\alpha_0| - |\beta_0|]|z|$$

for $R \ge 1$ and

$$|F(z)| \ge |a_0| - R[|a_n| + L + M - |\alpha_0| - |\beta_0|]|z|$$

for $R \le 1$.

Thus for $R \ge 1$, $|F(z)| > 0$ if $|z| < \dfrac{|a_0|}{R^{n+1}[|a_n| + L + M - |\alpha_0| - |\beta_0|]}$

and for $R \le 1$, $|F(z)| > 0$ if $|z| < \dfrac{|a_0|}{R[|a_n| + L + M - |\alpha_0| - |\beta_0|]}$.

In other words, all the zeros of F(z) lie in $|z| \ge \dfrac{|a_0|}{R^{n+1}[|a_n| + L + M - |\alpha_0| - |\beta_0|]}$ for $R \ge 1$ and in

$|z| \ge \dfrac{|a_0|}{R[|a_n| + L + M - |\alpha_0| - |\beta_0|]}$ for $R \le 1$.

Since the zeros of F(z) are also the zeros of P(z), it follows that all the zeros of P(z) lie in

$|z| \ge \dfrac{|a_0|}{R^{n+1}[|a_n| + L + M - |\alpha_0| - |\beta_0|]}$ for $R \ge 1$ and in

$|z| \ge \dfrac{|a_0|}{R[|a_n| + L + M - |\alpha_0| - |\beta_0|]}$ for $R \le 1$.

Again , for $|z| \le R$, we have, by using the hypothesis

$$|F(z)| \le |a_n||z|^{n+1} + |a_0| + |\alpha_n - \alpha_{n-1}||z|^n + ... + |\alpha_1 - \alpha_0||z| + |\beta_n - \beta_{n-1}||z|^n + ...$$

$$+ |\beta_1 - \beta_0||z|$$

$$\le |a_n|R^{n+1} + |a_0| + |\alpha_n - \alpha_{n-1}|R^n + ... + |\alpha_1 - \alpha_0|R + |\beta_n - \beta_{n-1}|R^n + ...$$

$$+ |\beta_1 - \beta_0|R$$

$$\le |a_0| + R^{n+1}[|a_n| + |\alpha_n - \alpha_{n-1}| + ... + |\alpha_1 - \alpha_0| + |\beta_n - \beta_{n-1}| + ...$$

$$+ |\beta_1 - \beta_0|]$$

$$= |a_0| + R^{n+1}[|a_n| + L + M - |\alpha_0| - |\beta_0|]$$

for $R \ge 1$ and

for $R \le 1$,

$$|F(z)| \le |a_0| + R[|a_n| + L + M - |\alpha_0| - |\beta_0|].$$

Hence, by using Lemma 2 and the above observations, it follows that the number of zeros of F(z) and therefore P(z) in $\frac{|a_0|}{R^{n+1}[|a_n| + L + M - |\alpha_0| - |\beta_0|]} \le |z| \le \frac{R}{c}, c > 1,$ does not exceed $\frac{1}{\log c} \log \frac{|a_0| + R^{n+1}[|a_n| + L + M - |\alpha_0| - |\beta_0|]}{|a_0|}$ for $R \ge 1$ and the number of zeros of P(z) in $\frac{|a_0|}{R[|a_n| + L + M - |\alpha_0| - |\beta_0|]} \le |z| \le \frac{R}{c}, c > 1,$ does not exceed $\frac{1}{\log c} \log \frac{|a_0| + R[|a_n| + L + M - |\alpha_0| - |\beta_0|]}{|a_0|}$ for $R \le 1$.

That completes the proof of Theorem 2.

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