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EXPECTED NUMBER OF LEVEL CROSSINGS OF A RANDOM ORTHOGONAL POLYNOMIAL

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ABSTRACT

This paper provides number of zeros of a class of orthogonal polynomial which is a sequence of mutually independent, normally distributed random variables with mean zero and variance unity then the average number of zeros of the random orthogonal polynomial is asymptotic to $\sqrt{n/3}$.

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INTRODUCTION

Let $y = \sum_{k=0}^{n} y_k(w) b_k \Phi_k(t)$ be random polynomial such that $\{y_k(w)\}_{k=0}^{n}$ is a sequence of mutually independent, normally distributed random variables with mean zero and variance unity and (Φ_k (t) is a sequence of classical Gegenbauer polynomials such that $\{b_k \Phi(t)\}_{k=0}^{n}$ is a sequence of normalized orthogonal polynomials. Then, it is proved that the average number of zeros of the random polynomial is asymptotic to $\sqrt{n/3}$.

Let $y=\sum_{k=0}^{n} b_k y_k(w)\Phi_k(t)$, 0<w<1 be random polynomial, where $\{y_0, (w), y_1 (w), \ldots\ldots\ldots y_n(w)\}$ is a sequence of mutually independent, normally distributed random variables with mathematical expectation zero and variance one. Let $\{\Phi_0 (t), \ldots\ldots \Phi_1 (t)\}$ is a sequence of real valued polynomials (functions) and $(b_0, b_1\ldots..)$ is a sequence of real constants. J.E. Littlewoods and A.C. Offord [4,5,6] showed that, when $b_k=1$ and $\Phi_k(t)=t^k$, most of the equation of the form y=0, have at most $25(\log n)^2$ real zeros for large n. When $b_0=0$, $b_k=1$ for $k\neq 0$, and $\Phi_0(t)= \cos k(\cos^{-1}t)$, J.E.A., Dunnage [3] estimated the average number of zeros of the family of equations y=0 to be asymptotic to $2n/\sqrt{3}$ in the interval (-1,1).

It is interesting to observe that while t^k's are a set of functions monotonic in $(-\infty,0)$ and $[0, \infty]$, $\cos k(\cos^{-1} t)$, for each k, oscillates k times between –1 and 1. The fact that the average number of zeros of y=0 when $\Phi_k(t)=\cos k (\cos^{-1}t)$ is proportional to the number of individual oscillations of $\Phi_k(t)$ about the t-axis, draws attention to the equation as to how far the oscillatory nature of $\Phi_k(t)$ decisively affects the zeros of y=0. Although the answer remains still inconclusive, we attempt to show that for large n, the above equation may be excepted to have c.n., (C>0) number of real roots when $\Phi_k(t)$ happens to be the ultra spherical classical orthogonal polynomial (Gegenbauer polynomial). In other words the oscillatory property of $\Phi_k(t)$ is also shared by $\sum_{k=0}^{n} b_k y_k(w)\Phi_k(t)$

Now $\Phi_k(t)$ is associated with a weight function $u(t)=(1-t^2)^{-1/2}$, $\lambda>1/2$ corresponding to the interval (-1,1) over which the integral of u(t), $\Phi_k(t)$ is a positive number h_k. We take $b_k=h^{-1/2}{}_k$. Then the integral of $\Psi^2{}_k(t)=b^2{}_k\Phi^2{}_k(t)$ over the given interval is unity, so that each of the terms of the polynomial $\sum_{k=0}^{n} b_k y_k(w)\Psi_k(t)=\sum_{k=0}^{n} b_k y_k(w)\Phi_k(t)$ has same weightage in the same sense.

Thus, in what follows, we find the average number of zeros of the equation

$$\sum_{k=0}^{n} b_k y_k(w)\Psi_k(t) \qquad (1.1)$$

We denote by ENn(f:α,β) the expected number of real zeros of (1.1) in the interval (α,β). Das[2] was first to find ENn(f: α,β) for a random orthogonal polynomial, although $\Psi_k(t)$

considered by him was a normalized Legendre polynomial, which is a special case of the polynomial considered by us.

1.2. Formula for ENn (f: a,b) Following the procedure of Kac,[7] we obtain

$$EN_n(f:a,b)=\frac{1}{\pi}\int_a^b\frac{(\{X_n(t)\}\{Z_n(t)\}-\{Y_n(t)\}^2)^{1/2}}{X_n(t)}dt \tag{1.2}$$

where $X_n(t)=X_n\sum_{k=0}^{n}\Psi_k(t)^2$

$$Y_n(t)=Y_n\sum_{k=0}^{n}[\Psi'_k(t)][\Psi'_k(t)]$$

$$Z_n(t)=Z_n\sum_{k=0}^{n}[\Psi'_k(t)]^2$$

provided that $X_nZ_n\text{-}Y_n{}^2>0.$

The last inequality holds good by Cauchy's inequality.

Let us put $\mu_n=I_n h_n{}^{-1}r^{-1}{}_{n+1}$ where r_n is the coefficient of t^n in $\Phi_n(t)$. The famous Crammer and Leadbetter[1] formula of the theory of orthogonal functions reads as follows.

$$\sum_{k=0}^{n}h^{-1}{}_k\Phi_k(\mu)\Phi_k(t)=\mu_n\frac{\{\Phi_{n+1}(\mu)\Phi_n(t)-\Phi_n(\mu)\Phi_{n+1}(t)\}}{\mu-t} \tag{1.3}$$

Putting $\mu=t+\gamma$ in the formula (1.3), we obtain by Taylor's expansion that

$$\sum_{k=0}^{n}h^{-1}{}_k\Phi_k(\mu)\Phi_k(t+\gamma)=\mu_n\frac{\{\Phi_{n+1}(\mu)\Phi_{n+1}(t)-\Phi_n(\mu)\Phi_{n+1}(t+\gamma)\Phi_{n+1}(t)\}}{t+\gamma-t}$$

Now

$$LHS(=)\sum_{k=0}^{n}h^{-1}{}_k(t)\Phi_k(t)+\gamma\Phi'_k(t)+\frac{\gamma^2}{2!}\Phi''_k(t)+......$$

$$=\sum_{k=0}^{n}h^{-1}{}_k\Phi^2{}_k(t)+\gamma\sum_{k=0}^{n}h^{-1}{}_k\Phi_k(t)+\Phi'_k(t)+\frac{\gamma^2}{2!}\sum_{k=0}^{n}h^{-1}{}_k\Phi_k(t)+\Phi''_k(t)+......$$

$$RHS_n(=)\mu\left[\Phi_n(t)\left\{\Phi_{n+1}(t)+\gamma\Phi'_{n+1}(t)++\gamma\Phi'_k(t)+\frac{\gamma^2}{2!}\Phi''_{n+1}(t)+\frac{\gamma^3}{3!}\Phi'''_{n+1}(t)\right\}\right]$$

$$-\Phi_{n+1}(t)\left\{\Phi_n(t)+\gamma\Phi'_n(t)+\frac{\gamma^2}{2!}\Phi''_n(t)+\frac{\gamma^3}{3!}\Phi'''_n(t)\right\}$$

$$=\mu_n\left\{\Phi_n(t)+\Phi'_{n+1}(t)+\tfrac{1}{2}\Phi_n(t)+\Phi''_{n+1}(t)\gamma+\frac{1}{6}\Phi_n(t)\Phi''_{n+1}(t)\gamma^2\right\}$$

$$-\Phi_{n+1}(t)\Phi'_n(t)-\tfrac{1}{2}\Phi_{n+1}(t)\Phi''_n(t)\gamma-\frac{1}{6}\Phi_{n+1}(t)\Phi'''_n(t)\gamma^2....$$

Now, equating coefficients of like powers of γ on both the sides, we obtain

$$\sum_{k=0}^{n} h_k^{-1}[\Phi_k(t)] = \mu_n[\Phi'_{n+1}(t)\Phi_n(t) - \Phi_{n+1}(t)\Phi'_n(t)], \quad (1.4)$$

$$\sum_{k=0}^{n} h_k^{-1}[\Phi_k(t)\Phi'_k(t)] = \frac{\mu_n}{2}[\Phi''_{n+1}(t)\Phi_n(t) - \Phi_{n+1}(t)\Phi''_n(t)], \quad (1.5)$$

$$\sum_{k=0}^{n} h_k^{-1}[\Phi_k(t)\Phi'_k(t)] = \frac{\mu_n}{3}[\Phi''_{n+1}(t)\Phi_n(t) - \Phi_{n+1}(t)\Phi'''_n(t)], \quad (1.6)$$

Differentiating (1.5), we get

$$\sum_{k=0}^{n} h_k^{-1}[\Phi'_k(t) + \Phi''_k(t)] =$$

$$\frac{\mu_n}{2}[\Phi'''_{n+1}(t)\Phi_n(t) + \Phi'_n(t)\Phi''_{n+1}(t) - \Phi'_{n+1}(t)\Phi''_n(t) - \Phi'''_n(t)\Phi_{n+1}(t)],$$

$$\sum_{k=0}^{n} h_k^{-1}[\Phi'_k(t) + \Phi''_k(t)] =$$

$$\frac{\mu_n}{2}[\Phi'''_{n+1}(t)\Phi_n(t) + \Phi'_n(t)\Phi''_{n+1}(t) - \Phi'_{n+1}(t)\Phi''_n(t) - \Phi_n(t)\Phi'''_n(t)]$$

or

$$\sum_{k=0}^{n} h_k^{-1}[\Phi'_k(t)]^2 =$$

$$\frac{\mu_n}{2}[\Phi''_{n+1}(t)\Phi'_n(t) - \Phi'_{n+1}(t)\Phi''_n(t)]$$ Hence, from (1.4), (1.5) and (1.7), it is evident that

$$+\frac{\mu_n}{6}[\Phi'''_{n+1}(t)\Phi_n(t) - \Phi_{n+1}(t)\Phi'''_n(t)] (1.6)$$

$$\sum_{k=0}^{n} h_k^{-1}[\Phi_k(t)]^2 = \sum_{k=0}^{n} h_k^{-1/2}[\Phi_k(t)]^2$$

$$= \sum_{k=0}^{n} (\Psi_k(t))^2 = X_n$$

and

$$\sum_{k=0}^{n} h_k^{-1}[\Phi_k(t)]^2 = \sum_{k=0}^{n} h_k^{-1/2}[\Phi_k(t)]^2$$

$$= \sum_{k=0}^{n} (\Psi'_k(t))^2 = Y_n$$

and

$$\sum_{k=0}^{n} h_k^{-1}[\Phi_k(t)]^2 = \sum_{k=0}^{n} h_k^{-1/2}[\Phi_k(t)]^2$$

$$= \sum_{k=0}^{n} (\Psi_k^{'}(t))^2 = Z_n$$

Now, making use of (1.4), (1.5) and (1.7), together with the fact that $\mu_n \neq 0$, we obtain

$$\frac{X_n Z_n - Y^2_n}{X^2_n} = \frac{Z_n}{X_n} - \left(\frac{Y_n}{X_n}\right)^2$$

$$= \left\{ \frac{\frac{\mu_n}{2}\left[\Phi^{''}_{n+1}(t)\Phi^{'}_{n}(t) - \Phi^{'}_{n+1}(t)\Phi^{''}_{n}(t)\right] + \frac{\mu_n}{6}\left[\Phi^{'''}_{n+1}(t)\Phi_n(t) - \Phi_{n+1}(t)\Phi^{'''}_{n}(t)\right]}{\mu_n\left[\Phi^{'}_{n+1}(t)\Phi_n(t) - \Phi_{n+1}(t)\Phi^{'}_{n}(t)\right]} \right\}^2 \text{ i.e. } EN_n(f:a,b) = \frac{1}{\pi}\int_a^b g_n(t)dt. \quad (1.8)$$

where

$$g^2_n(t) = \left[\frac{W_n(t) + V_n(t)}{R_n(t)} - \frac{U^2_n(t)}{4R_n^2(t)}\right],$$

$$R_n(t) = R_n = \Phi^{'}_{n+1}(t)\Phi_n(t) - \Phi_{n+1}(t)\Phi^{'}_{n}(t),$$

$$U_n(t) = U_n = \Phi^{''}_{n+1}(t)\Phi_n(t) - \Phi_{n+1}(t)\Phi^{''}_{n}(t),$$

$$V_n(t) = V_n = \tfrac{1}{2}\left[\Phi^{''}_{n+1}(t)\Phi^{'}_{n}(t) - \Phi^{'}_{n+1}(t)\Phi^{''}_{n}(t)\right]$$

and

$$W_n(t) = W_n = \tfrac{1}{6}\left[\Phi^{'''}_{n+1}(t)\Phi_n(t) - \Phi_{n+1}(t)\Phi^{'''}_{n}(t)\right]$$

1.3. Proof of the theorem

To prove the theorem, we divide the interval (-1,1) into three subintervals;

(i) (-1, ε, 1+ ε),

(ii) (-1, -1+ε), (iii). (1-ε, 1).

We choose $\in = n^{-\frac{1}{4+\delta}}$.

In the above section we find out the average number of zeros in the interval (-1, ε, 1+ ε).

In the section 1.4 we prove that the number of zeros in the intervals (ii) and (iii) are negligible in comparison to those in the interval (i).

1.4. Expected Number of Zeros in The Interval (-1,ε,1+ ε)

In order to evaluate EN_n(f: -1+ε, 1+ ε), we se the formula derived in 1.2.

From above , we have

$$(1-t^2)\Phi''_n(t) = (2\lambda+1)t\Phi'_n(t) + n(n+2\lambda)\Phi_n(t) \qquad (1.9)$$

and

$$(1-t^2)\Phi''_{n+1}(t) - (2\lambda+1)t\Phi'_{n+1}(t) - (n+1)(n+1+2\lambda)\Phi_{n+1}(t) \qquad (1.10)$$

Multiplying (1.9) by $\Phi''_{n+1}(t)$ and (1.10) by $\Phi''_n(t)$, we obtain

$$(1-t^2)\Phi''_n(t)\Phi'_{n+1}(t) = (2\lambda+1)t\Phi'_n(t)\Phi'_{n+1}(t) - n(n+2\lambda\lambda)_n(t)\Phi'_{n+1}(t) \qquad (1.11)$$

and

$$(1-t^2)\Phi''_{n+1}(t)\Phi'_n(t) = (2\lambda+1)t\Phi'_n(t)\Phi'_{n+1}(t) - n+1(n+1+2\lambda\lambda)_{n+1}(t)\Phi'_n(t) \qquad (1.12)$$

Subtracting (1.11) from (1.12), we have

$$(1-t^2)\Phi''_n(t)\Phi'_{n+1}(t) = (2\lambda+1)t\Phi'_n(t)\Phi'_{n+1}(t)$$
$$-n(n+2\lambda\lambda)_n(t)\Phi'_{n+1}(t) \qquad (1.13)$$

Multiplying (1.9) by $\Phi''_n(t)$ and (1.10) by $\Phi_n(t)$, we have

$$(1-t^2)\Phi''_n(t)\Phi'_{n+1}(t) = (2\lambda+1)t\Phi'_n(t)\Phi'_{n+1}(t)$$
$$-n(n+2\lambda\lambda)_n(t)\Phi'_{n+1}(t)(1.14)$$

and

$$(1-t^2)\Phi''_n(t)\Phi'_{n+1}(t) = (2\lambda+1)t\Phi'_n(t)\Phi'_{n+1}(t)$$
$$-n(n+2\lambda\lambda)_n(t)\Phi'_{n+1}(t) \qquad (1.15)$$

Subtracting (1.15) from (1.14), we have

$$(1-t^2)\Phi''_{n+1}(t)\Phi_n(t) - \Phi''_n(t)\Phi_{n+1}(t)$$
$$= (2\lambda+1)t\left(\Phi'_{n+1}(t)\Phi_n(t) - \Phi'_n(t)\Phi_{n+1}(t)\right)$$
$$-(2n+1+2\lambda\lambda)_{n+1}(t)\Phi_n(t) \qquad (1.16)$$

Differentiating (1.9) and (1.10), we obtain

$$-2t\Phi''_n(t) + (1-t^2)\Phi'''_n(t) = (2\lambda+1)\Phi'_n(t) + (2\lambda+1)t\Phi''_n(t) - n(n+2\lambda\lambda)'_n(t) \qquad (1.17)$$

and

$$-2t\Phi''_n(t)+(1-t^2)\Phi'''_n(t)=(2\lambda+1)\Phi'_{n+1}(t)$$
$$=(2\lambda+1)t\Phi'_{n+1}(t)+(2\lambda+1)\Phi'''_{n+1}(t) \quad (1.18)$$
$$-(n+1)\ (n+12\lambda)\Phi'_{n+1}(t)$$

Multiplying (1.9) by $\Phi''_n(t)$ and (1.10) by $\Phi_n(t)$, and subtracting, we have

$$(1-t^2)(\Phi'''_{n+1}(t)\Phi_n(t)-\Phi'''_n(t)\Phi_{n+1}(t)$$
$$=2t\left(\Phi'_{n+1}(t)\Phi_n(t)-\Phi_{n+1}(t)\Phi''_n(t)\right)$$
$$+(2\lambda+1)t(\Phi''_{n+1}(t)\Phi_n(t)-\Phi''_n(t)\Phi_{n+1}(t)$$
$$-n(n+2\lambda)(\Phi''_{n+1}(t)\Phi_n(t)-\Phi''_n(t)\Phi_{n+1}(t)$$

$$=(2\lambda+3)t(\Phi''_{n+1}(t)\Phi_n(t)-\Phi''_n(t)\Phi_{n+1}(t)$$
$$+(2\lambda+1)-n(n+2\lambda)\ (\Phi''_{n+1}(t)\Phi_n(t)-\Phi''_n(t)\Phi_{n+1}(t)$$
$$-(2n+1+2\lambda)\ (\Phi'_{n+1}(t)\Phi_n(t)$$

$$(2\lambda+3)t\left[\frac{(2\lambda+1)}{(1-t^2)}(\Phi'_{n+1}(t)\Phi_n(t)-\Phi'_n(t)\Phi_{n+1}(t)\right]$$
$$-\left[\frac{(2n+1+2\lambda)}{(1-t^2)}(\Phi'_{n+1}(t)\Phi_n(t)\right]$$
$$+\left[(2\lambda+1)-n(n+2\lambda)(\Phi'_{n+1}(t)\Phi_n(t)-\Phi'_n(t)\Phi_{n+1}(t)\right] \textit{(we have substituted)}$$
$$+\left[\frac{(2n+1+2\lambda)}{(1-t^2)}(\Phi_n(t)(n+1)t\Phi_{n+1}(t)-(2\lambda+1)\Phi_n(t)\right]$$

$$(t^2-1)(\Phi'_{n+1}(t)=(n+1)t\Phi_{n+1}(t)-(2\lambda+1)\Phi_n(t)$$
$$=\left[\frac{(2\lambda+3)(2\lambda+1)t^2}{(1-t^2)}+(2\lambda+1)-n(n+2\lambda)(\Phi'_{n+1}(t)-\Phi'_n(t)\Phi_{n+1}(t)\right]$$
$$-\left[\frac{(2n+1+2\lambda)(2\lambda+3)t}{(1-t^2)}-\frac{t(n+1)(2n+1+2\lambda)}{(1-t^2)}\right]\Phi_{n+1}(t)\Phi_n(t)$$
$$-\left[\frac{(2n+1+2\lambda)(2\lambda+1)\Phi_n^2(t)}{(1-t^2)}\right] \quad (1.19)$$

For large n, we shall use the asymptotic estimate of $\Phi_n(t)$ as

$$\Phi_n(t) \sim \frac{2^{\lambda}}{(\pi n)^{1/2}}(1-t)^{-\frac{1}{2}}(1+t)^{-\frac{1}{2}}\left[\cos x + o\left(\frac{1}{n\sin\theta}\right)\right],$$

where X=($n\theta+\lambda\theta-\lambda\theta$) and t-=cosθ. (We have taken $\alpha=\beta=\lambda-\frac{1}{2}$ From above, we get

$$(1-t^2)\Phi'_n(t) = (2\lambda-1+n)\Phi_{n-1}(t) - nt\Phi_n(t) \qquad (1.20)$$

and

$$(1-t^2)\Phi'_{n+1}(t) = (2\lambda+n)\Phi_n(t) - (n+1)t\Phi_{n+1}(t) \qquad (1.21)$$

From the two relations, we have

$$(1-t^2)(\Phi'_{n+1}(t)\Phi_n(t) - \Phi'_n(t)\Phi_{n+1}(t)$$

$$=(1-t^2)R_n(t)$$

$$=(\lambda+n)\Phi^2_n(t) - \Phi_{n+1}(t)\Phi_n(t) - (2\lambda+1+n)\Phi_{n-1}(t)\Phi_{n+1}(t).$$

From the two relations, we have

$$(1-t^2)(\Phi'_{n+1}(t)\Phi_n(t) - \Phi'_n(t)\Phi_{n+1}(t)$$

$$=(1-t^2)R_n(t) \qquad (1.22)$$

$$=(\lambda+n)\Phi^2_n(t) - \Phi_{n+1}(t)\Phi_n(t) - (2\lambda+1+n)\Phi_{n-1}(t)\Phi_{n+1}(t).$$

Hence

$$(1-t^2)R_n(t)$$

$$-(\lambda+n)\frac{2^{2\lambda}}{\pi n}(1-t)^{-\lambda}(1+t)^{-1}\left[\cos^2 X + O\left(\frac{1}{n\sin n\theta}\right)\right]$$

$$-t\frac{2^{2\lambda}}{\pi[n(n+1)]^{1/2}}(1-t)^{-\lambda}(1+t)^{-1}\left[\cos X\cos(X+\theta) + O\left(\frac{1}{n\sin n\theta}\right)\right]$$

$$-(2\lambda-1+n)\frac{2^{2\lambda}}{\pi[n(n+1)]^{1/2}}(1-t)^{-\lambda}(1+t)^{-1}\left[\cos X\cos(X-\theta) + O\left(\frac{1}{n\sin n\theta}\right)\right]$$

$$\sim \sigma^2\left[\cos^2 X + O\left(\frac{1}{n\sin n\theta}\right)\right] - \sigma^2\left[\cos X\cos(X-\theta)\cos(X+\theta) + O\left(\frac{1}{n\sin n\theta}\right)\right]$$

$$\sigma^2\left[\sin^2\theta + O\left(\frac{1}{n\sin n\theta}\right)\right]$$

where

$$\sigma = \frac{2^{\lambda}}{\pi^{1/2}}(1-t)^{-\lambda/2}(1+t)^{-\lambda/2}.$$

Hence

$$(1-t^2)R_n(t)$$

$$= \frac{2^{2\lambda}}{\pi n}(1-t)^{-\lambda}(1+t)^{-1}\left[(1-t^2)+O\left(\frac{1}{n\sin n\theta}\right)\right] \quad (1.23)$$

$$R_n(t) = \frac{2^{2\lambda}}{\pi(1-t^2)}(1-t)^{-\lambda}(1+t)^{-1}\left[(1-t^2)+O\left(\frac{1}{n\sin n\theta}\right)\right]$$

Now

$$\Phi_n(t)\Phi_{n+1}(t) = \frac{2^{2\lambda}}{\pi n}(1-t)^{-\lambda}(1+t)^{-1}\left[\cos X\cos(X+\theta)+O\left(\frac{1}{n\sin n\theta}\right)\right]$$

$$\le \frac{K}{n}(1-t)^{-\lambda}(1+t)^{-1} \quad (1.24)$$

and

$$\Phi^2{}_n(t) \le \frac{K}{n}(1-t)^{-\lambda}(1+t)^{-1} \quad (1.25)$$

Hence

$$\frac{\Phi_n(t)\Phi_{n+1}(t)}{\Phi'_{n+1}(t)\Phi_n(t)-\Phi'_n(t)\Phi_{n+1}(t)} \le \frac{\frac{K}{n}(1-t)^{-\lambda}(1+t)^{-\lambda}}{(1-t^2)^{-1}(1+t)^{-\lambda}\left\{(1-t^2)+O\left(\frac{1}{n\sin n\theta}\right)\right\}}$$

So that

$$\frac{\Phi_{n+1}(t)\Phi_n(t)}{R_n(t)} = O\left(\frac{1}{n}\right) \quad (1.26)$$

and

$$\frac{\Phi^2{}_n(t)}{R_n(t)} = O\frac{1}{n} \quad (1.27)$$

Now

$$(1-t^2)\Phi'_n(t)\Phi_{n+1}(t)$$

$$= (2\lambda-1+n)\Phi'_{n-1}(t)\Phi_{n+1}(t) - nt\Phi_n(t)\Phi_{n+1}(t)$$

$$= (2\lambda-1+n)\frac{\sigma^2}{n}\left[\cos(X-\theta)\cos(X+\theta)+O\left(\frac{1}{n\sin n\theta}\right)\right]$$

$$- nt\frac{\sigma^2}{n}\left\{\cos X+O\left(\frac{1}{n\sin n\theta}\right)\right\}\left\{\cos(X+\theta)+O\left(\frac{1}{n\sin n\theta}\right)\right\}.$$

Hence

$$\frac{\Phi'_n(t)\Phi_{n+1}(t)}{R_n(t)} = O\left(\frac{1}{(1-t^2)}\right)(1.28)$$

Now

$$\frac{V_n(t)}{R_n(t)} = \frac{\Phi''_{n+1}(t)\Phi'_n(t) - \Phi'_{n+1}(t)\Phi''_n(t)}{2\left(\Phi'_{n+1}(t)\Phi_n(t) - \Phi_{n+1}(t)\Phi'_n(t)\right)}$$

$$= \frac{\{n(n+2\lambda)(\Phi_n(t)\Phi'_{n+1}(t) - \Phi'_n(t)\Phi_{n+1}(t)) - (2n+1+2\lambda)\Phi_{n+1}(t)\Phi'_n(t)\}}{2\left(\Phi'_{n+1}(t)\Phi_n(t) - \Phi_{n+1}(t)\Phi'_n(t)\right)}$$

$$= \frac{n(n+2\lambda)}{2(1-t^2)} - \frac{\{2n+1+2\lambda\}}{2(1-t^2)}\frac{\{\Phi_{n+1}(t)\Phi'_n(t)\}}{\left(\Phi'_{n+1}(t)\Phi_n(t) - \Phi_{n+1}(t)\Phi'_n(t)\right)} \quad = \frac{(2\lambda+3)(2\lambda+1)t^2}{6(1-t^2)^2} + \frac{\{1+2\lambda\}}{6(1-t^2)} - \frac{n\{n+2\lambda\}}{6(1-t^2)}$$

$$= \frac{n^2}{2(1-t^2)} + O\left(\frac{n}{2(1-t^2)}\right), \quad (1.29) \qquad = \frac{(2n+1+2\lambda)(2\lambda+3)}{(1-t^2)^2} + \frac{\Phi^2_n(t)}{6\left(\Phi'_{n+1}(t)\Phi_n(t) - \Phi_{n+1}(t)\Phi'_n(t)\right)}$$

$$= \frac{n^2}{6(1-t^2)} + O\left(\frac{n}{(1-t^2)^2}\right)(1.30),$$

From (1.28) and (1.29), we have

$$\frac{W_n(t)+V_n(t)}{R_n(t)} = \frac{n^2}{3(1-t^2)} + O\left(\frac{n}{(1-t^2)^2}\right), \qquad (1.31)$$

Also we have

$$\frac{U_n(t)}{2R_n(t)} = \frac{\Phi''_{n+1}(t)\Phi_n(t) - \Phi_{n+1}(t)\Phi''_n(t)}{2\left(\Phi'_{n+1}(t)\Phi_n(t) - \Phi_{n+1}(t)\Phi'_n(t)\right)}$$

$$= \left\{\frac{(2\lambda+1)t(\Phi'_{n+1}(t)\Phi_n(t) - \Phi'_n(t)\Phi_{n+1}(t)) - (2n+1+2\lambda)\Phi_{n+1}(t)\Phi_n(t)}{2(1-t^2)(\Phi'_{n+1}(t)\Phi_n(t) - \Phi'_n(t)\Phi_{n+1}(t))}\right\}$$

$$= \frac{(2\lambda+1)t}{2(1-t^2)^2} - \frac{\{2n+1+2\lambda\}}{2(1-t^2)}\frac{\Phi_{n+1}(t)\Phi_n(t)}{\left(\Phi'_{n+1}(t)\Phi_n(t) - \Phi_{n+1}(t)\Phi'_n(t)\right)}$$

$$= \quad = \frac{(2\lambda+3)(2\lambda+1)t^2}{6(1-t^2)^2} + \frac{\{1+2\lambda\}}{6(1-t^2)} - \frac{n\{n+2\lambda\}}{6(1-t^2)} \quad O\left(\frac{1}{(1-t^2)^2}\right) + O\left(\frac{1}{n(1-t^2)^2}\right).$$

$$= \frac{(2n+1+2\lambda)(2\lambda+3)}{(1-t^2)^2} + \frac{\Phi^2_n(t)}{6\left(\Phi'_{n+1}(t)\Phi_n(t) - \Phi_{n+1}(t)\Phi'_n(t)\right)}$$

Hence

$$\frac{W_n(t)+V_n(t)}{R_n(t)}\frac{U^2_n(t)}{4R^2_n(t)} = \frac{n^2}{3(1-t^2)} + O\left(\frac{n}{(1-t^2)^2}\right) \quad (1.32)$$

and

$$\frac{U_n(t)}{4R^2{}_n(t)} = O\left(\frac{1}{(1-t^2)^2}\right) \quad (1.33)$$

so that

$$g_n(t) = \sqrt{\frac{W_n(t)+V_n(t)}{R_n(t)} - \frac{U^2{}_n(t)}{4R^2{}_n(t)}} = \frac{n}{\sqrt{3}(1-t^2)^{1/2}} + O\left(\frac{1}{(1-t^2)^2}\right)^{1/2}$$

For the range (-1+ε, 1- ε) , we notice that

$$1-t^2 > 2\in - \in^2 = \frac{2n^{\frac{1}{4+\delta}} - 1}{n^{\frac{2}{4+\delta}}}$$, where $\in = n^{\frac{1}{4+\delta}}$, as previously specified.

Thus $(1-t^2)^{-1} = O\left(n^{\frac{1}{4+\delta}}\right)$.

This observation together with (1.33), shows that

$$g_n(t) = \frac{n}{\sqrt{3}(1-t^2)^{1/2}}\left[1+O\left(n^{-\frac{2+\delta}{4+\delta}}\right)\right]^{1/2}$$

$$= \frac{n}{\sqrt{3}(1-t^2)^{1/2}}\left[1+O\left(n^{-\frac{2+\delta}{4+\delta}}\right)\right] \quad (1.34)$$

Thus from (1.8), we have

EN_n(f: -1+ε, 1+ ε),

$$\int_{-1+\in}^{1-\in} \frac{n}{\sqrt{3}(1-t^2)^{1/2}}\left[1+O\left(n^{-\frac{2+\delta}{4+\delta}}\right)\right]$$

$$= \frac{n}{\sqrt{3}}\left[1+O\left(n^{-\frac{2+\delta}{4+\delta}}\right)\right]\left[\sin^{-1} t\right]_{-1+\in}^{1-\in}$$

$$= \frac{n}{\pi\sqrt{3}}\left[1+O\left(n^{-\frac{2+\delta}{4+\delta}}\right)\right]\left[\sin^{-1}(1-\in) - \sin^{-1}(\in -1)\right]$$

$$= \frac{n}{\pi\sqrt{3}}\left[1+O\left(n^{-\frac{2+\delta}{4+\delta}}\right)\right]\left[2\sin^{-1}(1-\in)\right]$$

$$= \frac{n}{\sqrt{3}}\left[1+O\left(n^{-\frac{2+\delta}{4+\delta}}\right)\right]$$

(as $\sin^{-1}(1-\in) \sim \pi/2$) (1.35)

1.5. Number of Zeros in Subintervals (ii) and (iii).

Here we show that in the range (1-ε,1) and (-1, -1+ε) the number of zeros of (1.1) is negligibly small in comparisons to $EN_n(f: -1+\varepsilon, 1+\varepsilon)$,

Let $f(z) = f(\vec{y}(w), z) = \sum_{k=0}^{n} y_k(w)\Psi_k(z)$ (1.36)

where y (w) denotes the random vector $(y_0(w), y_1(w), \ldots y_n(w))$.

Now $f(\vec{y}(w), 1) = \sum_{k=0}^{n} y_k(w)\Psi_k(1)$,

is a random variable with mean zero and

variance $\sigma^2 = \sum_{k=0}^{n} \Psi^2{}_k(1) \geq \Psi_0{}^2(1) \geq 0$,

and hence has the distribution function

$$\frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{t} \exp\left(-\frac{v^2}{2\sigma^2}\right) dv.$$

Now

$$\begin{aligned} P\left(|f(1)| \leq e^{-2n\epsilon}\right) &= \left(\frac{2}{\pi\sigma^2}\right)^{1/2} \int_0^{e^{-2f}} \exp\left(-\frac{V^2}{2\sigma^2}\right) \\ &= \left(\frac{2}{\pi\sigma^2}\right)^{1/2} e^{-2ne} < e^{-ne} \end{aligned} \quad (1.37)$$

Let

$$I_n = \max_{0 \leq k \leq n} \left(y_k(w)\right) \quad (1.38)$$

$$P(I_n \leq n) = P\left(\max_{0\leq k\leq n}(y_k(w))\right)$$

$$= P\left(\prod_{k=0}^{n} |y_k(w)| \leq n\right)$$

$$= \left(\prod_{k=0}^{n} |P(y_k(w)| \leq n\right)$$

$$= \left(\prod_{k=0}^{n} |1-P(y_k(w)| > n\right)$$

Then
$$= \left(\prod_{k=0}^{n}\left[1-\sqrt{\frac{2}{\pi}\int_{n}^{\infty} e^{-v^2/2}dv}\right]\right)$$

$$\geq \left[1-(n+1)\sqrt{\frac{2}{\pi}\int_{n}^{\infty} e^{-v^2/2}dv}\right]1-e^{-n^{2/2}} \qquad (n > n_0). \qquad (1.39)$$

Let $T_n = \max_{0\leq k\leq n}\left|\Psi_k(1+2\in e^{i\theta})\right|$

For the Gegenbauer polynomials, h_n is determined

As $\dfrac{2^{1-2\lambda}\pi\Gamma(n+2\lambda)}{(n+\lambda)(\Gamma(\lambda)^2\Gamma(n+1)}$ for $\lambda > 1/2$.

Hence

$$b_n = h^{-1/2}{}_n < \alpha_1 n^{1/2}$$

where α_1 is a constant.

For the integral representation of Gegenbauer polynomial, we have

$$\Phi_n(t) = \frac{2^{1-2\lambda}\pi\Gamma(n+2\lambda)}{(n!(\Gamma(\lambda))^2}\int_0^n (t+i\sqrt{1-t^2}\cos\theta)n(\sin\theta)^{2\lambda-1}d\theta (1.40)$$

Remembering that $\in = n^{-1/(4+\delta)}$ we have

$$\left|\Phi_n(1+2\in e^{i\theta}\right| < \frac{2^{1-2\lambda}\pi\Gamma(n+2\lambda)}{(n!(\Gamma(\lambda))^2}(1+2\in)^n$$

$$< \alpha_3 n^{\alpha 2}(1+2\in)^n$$

$$< \alpha_3 n^{\alpha 2}\exp(2n^{\frac{3+\delta}{4+\delta}}), \ (1.41)$$

where α_2 and α_3 are constants involving λ only.

Hence from (1.40), we get $T_n < An^{\alpha 2+1/2} \exp(2n^{\frac{3+\delta}{4+\delta}})$, (1.42)

where A is a constant

Also

$$\left| f(1+2\in e^{i\theta} \right| = \left| \sum_{k=0}^{n} y_k(w)\Psi_k(1+2\in e^{i\theta}) \right|$$

$$\leq \left| \sum_{k=0}^{n} y_k(w)\Psi_k(1+2\in e^{i\theta}) \right|$$

$$\leq l_n \sum_{k=0}^{n} T_n = nl_n T_n \qquad (1.43)$$

Hence from (1.39), it follows that

$$P\left|(1+2\in e^{i\theta}\right| \leq n^2 T_n) \geq 1-e^{-n2/2} \qquad (1.44))$$

This together with (1.42), gives

$$P\left| f(1+2\in e^{i\theta}\right| \leq An^{\alpha} \exp(2n^{\frac{3+\delta}{4+\delta}}) \geq 1-e^{-n2/2} \; (1.45)$$

where $\alpha = \alpha_2 + 5/2$.

So from (1.37) and (1.45), we obtain

$$P\frac{\left| f(1+2\in e^{i\theta}\right|}{f(1)} \leq An^{\alpha} \exp(2n^{\frac{3+\delta}{4+\delta}} + 2n\in)$$

$$\geq P\left| f(1+2\in e^{i\theta}\right| \leq An^{\alpha} \exp(2n^{\frac{3+\delta}{4+\delta}})$$

$$-P\left| f(1)\right| \leq e^{-2ne}$$

$$> 1 - e^{-n2/2} - e^{-ne}$$

$$> 1 - \frac{2}{n} \qquad (1.46)$$

Let n(e) denote the number of zeros of f(y(w),z)=0 inside the circle $|z-1| \leq \in$.

It is easy to see that the number of zeros of (3.1.1) inside the interval $1-e \leq t \leq 1$ does not exceed n(e).

By Jensen's theorem, we have

$$n(\in) \leq \frac{1}{2\pi \log 2} \int_{\theta}^{2\pi} \log \left| \frac{f(1+2 \in e^{i\theta}}{f(1)} \right| d\theta \qquad \text{for } f(1) \neq 0$$

$$\frac{1}{2\pi \log 2} \int_{\theta}^{2\pi} \log \left\{ An^{\alpha} \exp\left(2n^{\frac{3+\delta}{4+\delta}} \right) + 2n \in \right\} d\theta, \quad (1.47)$$

except for a set of measure at most 2/n, as evident from (1.46).

Thus from (1.47) and remarks made earlier, we obtain that the number of zeros of (1.1) in (1-e,1) is at most $O\left(n^{\frac{3+\delta}{4+\delta}} \right)$ with probability at least 1-2/n.

An identical result is obtainable for the number of zeros of (1.1) in (-1, -1+e), so that EN_n(f:-1+e, 1-e)= $O\left(n^{\frac{3+\delta}{4+\delta}} \right)$.

The above derivation together with the estimate of EN_n(f:-1+e, 1-e) in section 1.3. proves that EN_n(f:-1,1)= $\frac{n}{\sqrt{3}} + O\left(n^{\frac{3+\delta}{4+\delta}} \right)$.

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