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FUZZY ALGEBRAIC STRUCTURE IN Z-ALGEBRAS

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ABSTRACT

In this paper, we introduce the notion of Fuzzy Z-Subalgebra of a Z-algebra and investigate their properties. We describe how to deal with the Z-homomorphism of image and inverse image of fuzzy Z-Subalgebras. We have also proved that the Cartesian product of fuzzy Z-Subalgebras is a fuzzy Z-Subalgebra.

KEYWORDS: Z-algebra, Z-Subalgebra, Z-homomorphism, Level Z-Subalgebras, Fuzzy Z-Subalgebras, Cartesian product of Z-algebras

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INTRODUCTION

Imai and Iseki introduced two new classes of abstract algebras: BCK algebras and BCI algebras (Imai and Iseki, 1966; Iseki, 1980). It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. In 2017, (Chandramouleeswaran et al., 2017) introduced the concept of Z-algebras as a new structure of algebra based on propositional calculus. By Propositions 3.7 and 3.8 of (Chandramouleeswaran et al., 2017), the Z-algebra is not a generalization of BCK/BCI-algebras.

In 1965, (Zadeh, 1965) introduced the fundamental concept of a fuzzy set which is a generalization of an ordinary set. In 1971, (Rosenfeld , 1971) introduced the notion of fuzzy groups. Following the idea of fuzzy groups, in 1991 (Xi, 1991) introduced the notion of fuzzy

BCK-algebras. In 2015, (Christopher Jefferson and Chandramouleeswaran, 2015) applied fuzzy algebraic structures in BP-algebras. In this paper, we study the fuzzy subalgebraic structures in Z-algebras and investigate some of their properties.

Preliminaries

In this section we recall some basic definitions.

Definition 2.1: (Iseki and Tanaka, 1978) A BCK- algebra $(X,*,0)$ is a nonempty set X with constant 0 and a binary operation $*$ satisfying the following conditions:

(i) $(x * y) * (x * z) \leq (z * y)$

(ii) $x * (x * y) \leq y$

(iii) $x \leq x$

(iv) $x \leq y$ and $y \leq x \Rightarrow x=y$

(v) $0 \leq x \Rightarrow x=0$, where $x \leq y$ is defined by $x * y = 0$,for all x, y, z ∈ X.

Definition 2.2: (Iseki ,1980)A BCI-algebra $(X,*,0)$ is a nonempty set X with constant 0 and a binary operation $*$ satisfying the following conditions:

(i) $(x * y) * (x * z) \leq (z * y)$

(ii) $x * (x * y) \leq y$

(iii) $x \leq x$

(iv) $x \leq y$ and $y \leq x \Rightarrow x = y$

(v) $x \leq 0 \Rightarrow x = 0$, where $x \leq y$ is defined by $x * y = 0$, for all x, y, z ∈ X.

Definition 2.3: (Chandramouleeswaran et al., 2017**)** A Z-algebra $(X,*,0)$ is a nonempty set X with constant 0 and a binary operation $*$ satisfying the following conditions:

(Z1) $x * 0 = 0$

(Z2) $0 * x = x$

(Z3) $x * x = x$

(Z4) $x * y = y * x$ when $x \neq 0$ and $y \neq 0$ $\forall$ x, y ∈ X.

Definition 2.4: (Chandramouleeswaran et al., 2017**)** Let S be a nonempty subset of a Z-algebra X. Then, S is called Z-Subalgebra of X if $x * y \in S$ for all x, y ∈ S.

Definition 2.5: (Chandramouleeswaran et al., 2017**)** Let $(X,*,0)$ and $(Y,*',0')$ be two Z-algebras. A mapping $h:(X,*,0)\to(Y,*',0')$ is said to be a **Z-homomorphism of Z-algebras** if $h(x*y)=h(x)*'h(y)$ for all $x,y\in X$.

Definition 2.6: Let h be a Z-homomorphism from the Z-algebra $(X,*,0)$ to the Z-algebra $(Y,*',0')$. Then

1. h is called

i) a **Z-monomorphism** of Z-algebras if h is 1-1.

ii) an **Z-epimorphism** of Z-algebras if h is onto.

2. h is called an **Z-endomorphism** of Z-algebras if h is a mapping from $(X,*,0)$ into itself.

Note: If $h:(X,*,0)\to(Y,*',0')$ is a Z-homomorphism then $h(0)=0'$.

Definition 2.7: (Zadeh**,** 1965**)** Let X be a nonempty set. A fuzzy set A in X is characterized by a membership function $\mu_A(x)$ which associates with each point x in X, a real number in the interval [0,1] with the value of $\mu_A(x)$ at x representing the "grade of membership" of x in A.

That is, a fuzzy set A in X is characterized by a membership function $\mu_A:X\to[0,1]$.

Definition 2.8: (Zadeh**,** 1965**)** The intersection of two fuzzy sets A and B with respective membership functions $\mu_A(x)$ and $\mu_B(x)$ is a fuzzy set C , written as $C=A\cap B$, whose membership function is related to those of A and B defined by,

$\mu_{A\cap B}(x)=\mu_C(x)=\min\{\mu_A(x),\mu_B(x)\}$, *for all* $x\in X$ or, in abbreviated form $\mu_C=\mu_A\wedge\mu_B$.

Definition 2.9: (Das P S, 1981**)** Let A be a fuzzy set of X. For a fixed $t\in[0,1]$, the set U(A;t)= $\{x\in X\,|\,\mu_A(x)\geq t\}$ is called an upper level subset (upper level cut, upper t-level subset) of A.

Definition 2.10: (Das P S, 1981**)** Let A be a fuzzy set of X. For a fixed $t\in[0,1]$, the set $L(A;t)=\{x\in X\,|\,\mu_A(x)\leq t\}$ is called a lower level subset (lower level cut, lower t-level subset) of A.

Note: (i) If $t_1\leq t_2$, $U(A;t_2)\subseteq U(A;t_1)$ and $L(A;t_1)\subseteq L(A;t_2)$.

(ii) $U(A;t) \cup L(A;t) = X$ for all $t \in [0,1]$.

Definition 2.11: (Rosenfeld A, 1971**)** A fuzzy set A in X with a membership function μ_A is said to have the sup property if for any subset $T \subset X$ there exists $x_0 \in X$ such that $\mu_A(x_0) = \sup\limits_{t \in T} \mu_A(t)$.

Definition 2.12: (Rosenfeld A, 1971**)** Let h be a mapping from X into Y.

i) Let A be a fuzzy set in X with a membership function μ_A. Then the image of A under h, denoted by h(A) is the fuzzy set in Y with a membership function $\mu_{h(A)}$ defined by

$$\mu_{h(A)}(y) = \begin{cases} \sup\limits_{z \in h^{-1}(y)} \mu_A(z) & \text{if } h^{-1}(y) = \{x \mid h(x) = y\} \neq \phi \\ 0 & , \text{ otherwise} \end{cases}$$

ii) Let B be a fuzzy set in Y with a membership function μ_B. The inverse image (or pre-image) of B under h, denoted by $h^{-1}(B)$ is the fuzzy set in X with a membership function $\mu_{h^{-1}(B)}$ defined by $\mu_{h^{-1}(B)}(x) = \mu_B(h(x))$ for all $x \in X$.

Definition 2.13: (Bhattacharya P and Mukherjee N P, 1985**)** Let A and B be the fuzzy sets of X and Y with a membership functions μ_A and μ_B respectively. Then, the Cartesian product $A \times B$ with membership function $\mu_{A\times B} : X \times Y \to [0,1]$ is defined as $\mu_{A\times B}(x, y) = \min\{\mu_A(x), \mu_B(y)\}$ for all $x \in X$ and $y \in Y$.

Definition 2.14: (Bhattacharya P and Mukherjee N P, 1985**)** Let A and B be the fuzzy sets of a set X with a membership functions μ_A and μ_B respectively. Then, the Cartesian product $A \times B$ with membership function $\mu_{A\times B} : X \times X \to [0,1]$ is defined as $\mu_{A\times B}(x, y) = \min\{\mu_A(x), \mu_B(y)\}$ for all $x, y \in X$.

Definition 2.15: (Bhattacharya P and Mukherjee N P, 1985**)** A fuzzy relation A on a nonempty set X is a fuzzy set A with a membership function $\mu_A : X \times X \to [0,1]$.

Definition 2.16: (Bhattacharya P and Mukherjee N P, 1985**)** If A is a fuzzy relation with a membership function μ_A on a set X and B is a fuzzy set of X with a membership function μ_B then A is a fuzzy relation on B if for all $x, y \in X$, $\mu_A(x, y) \leq \min\{\mu_B(x), \mu_B(y)\}$.

Definition 2.17(Bhattacharya P and Mukherjee N P , 1985**)** Let B be a fuzzy set on a set X with a membership function μ_B then the strongest fuzzy relation A_B on X, that is, a fuzzy relation A on B whose membership function $\mu_{A_B}: X\times X \to [0,1]$ is given by $\mu_{A_B}(x,y)=\min\{\mu_B(x),\mu_B(y)\}$.

Theorem 2.18: Let $(X,*,0)$ and $(Y,*',0')$ be two Z-algebras. Then $(X\times Y,*'',0'')$ is a Z-algebra where $(x_1,y_1)*''(x_2,y_2)=(x_1*x_2,y_1*'y_2)$ for all $(x_1,y_1),(x_2,y_2)\in X\times Y$, with $0''=(0,0')$ as constant element.

1. Fuzzy Z-Subalgebras in Z-algebras

In this section, we define the notion of Fuzzy Z-Subalgebra of a Z-algebra and prove some simple but elegant results.

Definition 3.1: Let $(X,*,0)$ be a Z-algebra. A fuzzy set A in X with a membership function μ_A is said to be a fuzzy Z- Subalgebra of a Z-algebra X if, for all x , y $\in X$ the following condition is satisfied : $\mu_A(x*y)\geq \min\{\mu_A(x),\mu_B(y)\}$.

Example 3.2: Let X= {0, 1, 2, 3} be a set with the following Cayley table:

$*$	**0**	**1**	**2**	**3**
0	0	1	2	3
1	0	1	3	2
2	0	3	2	1
3	0	2	1	3

Then $(X,*,0)$ is a Z-algebra.

Define a fuzzy set A in X with a membership function μ_A is given by

$$\mu_A(x)=\begin{cases}0.6 & \text{if} \quad x=0\\ 0.4 & \text{if} \quad x=1\\ 0.3 & \text{if} \quad x=\ 2,3\end{cases}$$

Then A is a fuzzy Z-Subalgebra of X.

Theorem 3.3: Intersection of any two fuzzy Z-Subalgebras of a Z-algebra X is again a fuzzy Z- Subalgebra.

Proof: Let A_1 and A_2 be fuzzy Z-Subalgebras of X. Let $x,y\in A_1\cap A_2$.

Then x, y $\in A_1$ and A_2. Since A_1 and A_2 are fuzzy Z-Subalgebras of X,

$$\mu_{A_1\cap A_2}(x*y) = \min\{\mu_{A_1}(x*y), \mu_{A_2}(x*y)\}$$
$$\geq \min\{\min\{\mu_{A_1}(x), \mu_{A_1}(y)\}, \min\{\mu_{A_2}(x), \mu_{A_2}(y)\}\}$$
$$= \min\{\min\{\mu_{A_1}(x), \mu_{A_2}(x)\}, \min\{\mu_{A_1}(y), \mu_{A_2}(y)\}\}$$
$$= \min\{\mu_{A_1\cap A_2}(x), \mu_{A_1\cap A_2}(y)\}$$

That is $\mu_{A_1\cap A_2}(x*y) \geq \min\{\mu_{A_1\cap A_2}(x), \mu_{A_1\cap A_2}(y)\}$

Hence $A_1 \cap A_2$ is a fuzzy Z – Subalgebras of X.

The above result can be generalized for a family of fuzzy Z-Subalgebras.

Corollary 3.4: Let $\{A_i | i \in \Omega\}$ be a family of fuzzy Z-Subalgebras of X. Then $\bigcap_{i\in\Omega} A_i$ is also a fuzzy Z-Subalgebra of X.

Theorem 3.5: A fuzzy set A of a Z-algebra X is a fuzzy Z-Subalgebra if and only if every $t \in [0,1]$, $U(A;t)$ is either empty or Z-Subalgebra of X.

Proof: Assume that A is a fuzzy Z-Subalgebra of a Z-algebra X and $U(A;t) \neq \phi$

To prove: $U(A;t)$ is a Z-subalgebra of X.

For any x , $y \in U(A;t)$, we have $\mu_A(x) \geq t$ and $\mu_A(y) \geq t$.

Then $\mu_A(x*y) \geq \min\{\mu_A(x), \mu_A(y)\}$

$\geq \min\{t, t\}$

$= t$

This implies $x*y \in U(A;t)$

That is, $U(A;t)$ is a Z-subalgebra of X.

Conversely, assume that $U(A;t)$ is a Z-Subalgebra of X.

To prove: A is a fuzzy Z-subalgebra of a Z-algebra X.

Let $x,y \in X$ and let $\mu_A(x) = t_1$ and $\mu_A(y) = t_2$. Then $x \in U(A;t_1)$ and $y \in U(A;t_2)$.

If $t_1 \leq t_2$, then $U(A;t_2) \subseteq U(A;t_1)$ and so $y \in U(A;t_1)$.

Since $U(A;t_1)$ is a Z-Subalgebra of X, $x*y \in U(A;t_1)$.

Thus $\mu_A(x*y) \geq t_1 = \min\{\mu_A(x), \mu_A(y)\}$, proving that A is a fuzzy Z-Subalgebra of X.

Definition 3.6: Let A be a fuzzy Z-Subalgebra of X. For any $t \in [0,1]$, Z-Subalgebras $U(A;t)$ are called Upper level Z-Subalgebras of A.

Remark 3.7: Henceforth, the Upper level Z-Subalgebras will be referred as level Z-Subalgebras.

Theorem 3.8: Any Z-Subalgebra of a Z-algebra X can be realized as a level Z-Subalgebra of some fuzzy Z-Subalgebra of X.

Proof: Let S be a Z-Subalgebra of a Z-algebra X and A be a fuzzy set in X defined by

$$\mu_A(x) = \begin{cases} t \text{ if } x \in S \\ 0 \text{ if } x \notin S \end{cases}$$

where $t \in [0,1]$ is fixed. Clearly U(A; t)=S.

To prove: A is a fuzzy Z- Subalgebra of a Z-algebra X.

We consider the following cases:

Case (i): If $x, y \in S$ then $x * y \in S$.

Hence $\mu_A(x) = \mu_A(y) = \mu_A(x * y) = t$ and

$\mu_A(x * y) \geq \min\{\mu_A(x), \mu_A(y)\}$.

Case (ii): If $x, y \notin S$ then $\mu_A(x) = \mu_A(y) = \mu_A(x * y) = 0.$

Then $\mu_A(x * y) \geq \min\{\mu_A(x), \mu_A(y)\} = 0.$

Case (iii): If at most one of x, $y \in S$ then atleast one of $\mu_A(x)$ and $\mu_A(y)$ is equal to 0.

Hence $\mu_A(x * y) \geq \min\{\mu_A(x), \mu_A(y)\} = 0.$

This shows that S is a level Z-Subalgebra of X corresponding to the fuzzy Z-Subalgebra A of X.

Theorem 3.9: Let X be a Z-algebra. Then given any chain of Z-Subalgebras $S_0 \subset S_1 \subset \cdots \subset S_r = X$, there exists a fuzzy Z-Subalgebra A of X whose upper t-level Z-Subalgebras are exactly the Z-Subalgebras of the chain.

Proof: Consider a set of numbers $t_0 > t_1 > t_2 > \cdots > t_r$ **,** where each $t_i \in [0,1]$.

Let $A : X \to [0,1]$ be a fuzzy set defined by $\mu_A(S_0) = t_0$ and $\mu_A(S_i - S_{i-1}) = t_i$, $i = 1,2,\ldots,r$.

Claim: A is a fuzzy Z-Subalgebra of X.

Let $x, y \in X$. Then we classify it into two cases as follows:

Case (1): Let $x, y \in S_i - S_{i-1}$. Then by the definition of A, $\mu_A(x) = t_i = \mu_A(y)$.

Since S_i is a Z-Subalgebra of X, it follows that $x*y \in S_i$ and so either $x*y \in S_i - S_{i-1}$ or $x*y \in S_{i-1}$. In any case, we conclude that $\mu_A(x*y) \geq t_i = \min\{\mu_A(x), \mu_A(y)\}$.

Case (2): For $i > j$, Let $x \in S_i - S_{i-1}$ and $y \in S_j - S_{j-1}$.

Then $\mu_A(x) = t_i$; $\mu_A(y) = t_j$ and $x*y \in S_i$, since S_i is a Z-Subalgebra of X and $S_j \subset S_i$.

Hence $\mu_A(x*y) \geq t_j = \min\{\mu_A(x), \mu_A(y)\}$.

Thus A is a fuzzy Z-Subalgebra of X.

From the definition of A, it follows that $\mathrm{Im}(A) = \{t_0, t_1, \cdots, t_r\}$.

Hence the upper t-level Z-Subalgebras of A are given by the chain of Z-Subalgebras.

$U(A;t_0) \subset U(A;t_1) \subset U(A;t_2) \subset \cdots \subset U(A;t_r) = X$.

Now $U(A;t_0) = \{x \in X \mid \mu_A(x_0) = t_0\} = S_0$.

Finally , we prove that $U(A;t_i) = S_i$ for $i = 1,2,...,r$.

Clearly $S_i \subseteq U(A;t_i)$.

If $x \in U(A;t_i)$, then $\mu_A(x) \geq t_i$ which implies that $x \not\in S_j$ for $j > i$.

Hence $\mu_A(x) \in \{t_1, t_2, \cdots, t_i\}$ and so $x \in S_k$ for some $k \leq i$.

As $S_k \subseteq S_i$, it follows that $x \in S_i \Rightarrow U(A;t_i) = S_i$ for $i = 1,2,...,r$.

This completes the proof.

Note: If X is a finite Z –algebra, then the number of Z-Subalgebras of X is finite whereas the number of level Z- Subalgebras of a fuzzy Z-Subalgebra A appears to be infinite. But since every level Z-Subalgebra is indeed Z-Subalgebra of X, not all these Z-Subalgebras are distinct. The next theorem characterizes this aspect.

Theorem 3.10: Let A be a fuzzy Z-Subalgebra of a Z-algebra X. Two level Z-Subalgebras $U(A;t)$ and $U(A;s)$ (with $t < s$) of A are equal if and only if there is no $x \in X$, $t \leq \mu_A(x) < s$.

Proof: Let A be a fuzzy Z-Subalgebra of a Z-algebra X.

Assume that $U(A;t) = U(A;s)$ for some $t < s$ and there exists $x \in X$ such that $t \leq \mu_A(x) < s$.

Then $U(A;s)$ is a proper subset of $U(A;t)$ which is a contradiction.

Hence there is no $x \in X$ such that $t \leq \mu_A(x) < s$.

Conversely, Suppose that there is no $x \in X$ such that $t \leq \mu_A(x) < s$. Since $t < s$, we get

$U(A;s) \subseteq U(A;t)$ (1)

If $x \in U(A;t)$ then $\mu_A(x) \geq t$ and so $\mu_A(x) > s$, because $\mu_A(x)$ does not lie between t and s.

Hence $x \in U(A;s)$.

Hence $U(A;t) \subseteq U(A;s)$ (2)

From (1) and (2) we get $U(A;t) = U(A;s)$.

Remark 3.11: As a consequence of **Theorem 3.10**, the level Z-Subalgebras of a fuzzy Z-Subalgebra A of a finite Z-algebra X form a chain and so we have the chain $U(A;t_0) \subset U(A;t_1) \subset \cdots \subset U(A;t_r) = X$, where $t_0 > t_1 > t_2 > ... > t_r$.

Corollary 3.12: Let X be a finite Z-algebra and A be a fuzzy Z-Subalgebra of X. If $Im(A) = \{t_1, \cdots, t_n\}$, then the family of Z-Subalgebras $U(A;t_i), i = 1,2,...,n$, constitutes all the level Z-Subalgebra of A.

Proof: Let $t \in [0,1]$ and $t \notin Im(A)$. Suppose $t_1 < t_2 < \cdots < t_n$ without loss of generality.

If $t \leq t_1$, then $U(A;t_1) = X = U(A;t)$.

If $t > t_n$, then $U(A;t) = \phi$ obviously.

If $t_{i-1} < t < t_i$, then $U(A;t) = U(A;t_i)$ by **Theorem 3.10**. Thus for any $t \in [0,1]$, the level Z-Subalgebra is one of $\{U(A;t_i) \mid i = 1,2,\cdots,n\}$.

Lemma 3.13: Let X be a Z-algebra and A be a fuzzy Z-Subalgebra of X. If Im(A) is finite, say $\{t_1,t_2,...,t_n\}$ then for any t_i, $t_j \in Im(A)$, $U(A;t_i) = U(A;t_j)$ implies $t_i=t_j$.

Proof: Assume that $t_i \neq t_j$ and $t_i < t_j$.

If $x \in U(A;t_j)$ then $\mu_A(x) \geq t_j > t_i$.

Hence $x \in U(A;t_i)$

Let $x \in X$ such that $t_i < \mu_A(x) < t_j$.

Then $x \in U(A;t_i)$ but $x \notin U(A;t_j)$

Hence $U(A;t_j) \subset U(A;t_i)$ and

$U(A;t_j) \neq U(A;t_i)$ a contradiction.

Then, $U(A;t_i) = U(A;t_j)$

Therefore $t_i = t_j$.

Theorem 3.14: Let A and B be two fuzzy Z-Subalgebras of a Z-algebra X with identical family of level Z-Subalgebras. If Im(A)={$t_1,t_2,...,t_r$} and Im(B)=$\{q_1, q_2,..., q_k\}$ where $t_1 \geq t_2 \geq ... \geq t_r$ and $q_1 \geq q_2 \geq ... \geq q_k$. Then

i) $k = r$

ii) U(A; t_i) = $U(B;q_i)$, $i = 1,2,\ldots,r$

iii) If $x \in X$ such that $\mu_A(x) = t_i$ then $\mu_B(x) = q_i$ $i = 1,2,\ldots,r$.

Proof: Let A and B be two fuzzy Z-Subalgebras of X with identical family of level Z-Subalgebras with F(A)=F(B) where F(A)= $\{U(A;t_i) | i = 1,2,...,r\}$ and F(B) = $\{U(B;q_i) | i = 1,2,..., k\}$.

Let Im(A) = {$t_1,t_2,...,t_r$} where $t_1 \geq t_2 \geq ... \geq t_r$ (1)

and let Im(B) = $\{q_1, q_2,..., q_k\}$ where $q_1 \geq q_2 \geq ... \geq q_k$ (2)

From (1) we get $U(A;t_1) \subseteq U(A;t_2) \subseteq ... \subseteq U(A;t_r) = X$ (3)

From (2) we get $U(B;q_1) \subseteq U(B;q_2) \subseteq ... \subseteq U(B;q_k) = X$ (4)

To prove (i): k = r

Suppose $k \neq r$, then consider the following cases:

Case (i): $k > r$

Let $k > r$ then U(A; t_i)= U(B; q_i) i=1,2,...,r

This shows that both t_i and q_i $\in$ Im(A)

For i > r we observe that $t_i \notin$ Im (A) and hence,

U(A; t_i) $\neq$ U(B; q_i), i= r+1, r+2,...,k.

Case (ii): $r > k$

Let $r > k$ then U(A; t_i) = U(B; q_i) i=1,2,...,k

This shows that both t_i and q_i $\in$ Im (B).

For i > k we observe that $q_i \notin$ Im (B) and hence

U(A; t_i) $\neq$ U(B; q_i) , i=k+1,k+2,...,r.

From (3) and (4) we get $t_i \neq q_i$ for all i=1,2,...,r.

Hence we can find some i such that U(A; t_i) $\neq$ U(B; q_i).

This contradicts that F(A)=F(B).

Hence we conclude that k = r.

To prove (ii): By part (i), we have proved that k = r. Since A and B have identical family of level Z-Subalgebras, we have

$U(A; t_i) = U(B; q_i)$, i=1,2,...,r.

To prove (iii): Let $x \in X$ such that $\mu_A(x) = t_i$ and $\mu_B(x) = q_j$

From (ii) follows that $x \in U(B; q_i)$, thus

$\mu_B(x) \geq q_i$ and $q_j \geq q_i$

Therefore $U(B; q_j) \subseteq U(B; q_i)$

Since $x \in U(B; q_j) = U(A; t_j)$, we get $t_i = \mu_A(x) \geq t_j$, this

gives $U(B; q_i) = U(A; t_i) \subseteq U(A; t_j) = U(B; q_j)$

Thus $U(B; q_j) = U(B; q_i)$ and by above **lemma:3.13** we get $q_j = q_i$.

Hence $\mu_B(x) = q_i$.

Hence the proof.

Corollary 3.15: Let A and B be two fuzzy Z-Subalgebras of X with identical family of level Z-Subalgebras. Then Im(A)=Im(B) implies A = B.

Proof: Let Im(A) = Im(B)= $\{q_1, q_2, \ldots, q_r\}$ where $q_1 \geq q_2 \geq \ldots \geq q_r$.

By **Theorem 3.14**, for any $x \in X$ there exists q_i such that $\mu_A(x) = q_i = \mu_B(x)$.

Thus $\mu_A(x) = \mu_B(x)$ for all $x \in X$.

This implies A=B.

4. Z -Homomorphism on Fuzzy Z-Subalgebras of Z-algebras:

In this section, we prove some simple theorems on fuzzy Z-Subalgebras under Z-homomorphisms in Z-algebras.

Theorem 4.1: Let h be a Z-homomorphism from a Z-algebra $(X, *, 0)$ onto a Z–algebra $(Y, *', 0')$ and let A be a fuzzy Z-Subalgebra of X with the supremum property. Then the image of A denoted by h(A) is a fuzzy Z-Subalgebra of Y.

Proof: Let $a, b \in Y$ with $x_0 \in h^{-1}(a)$ and $y_0 \in h^{-1}(b)$ such that $\mu_A(x_0) = \sup_{t \in h^{-1}(a)} \mu_A(t)$;

$\mu_A(y_0) = \sup_{t \in h^{-1}(b)} \mu_A(t)$.

$$\begin{aligned}
\mu_{h(A)}(a *' b) &= \sup_{t \in h^{-1}(a *' b)} \mu_A(t) \\
&\geq \mu_A(x_0 * y_0) \\
&\geq \min\{\mu_A(x_0), \mu_A(y_0)\} \\
&= \min\left\{ \sup_{t \in h^{-1}(a)} \mu_A(t), \sup_{t \in h^{-1}(b)} \mu_A(t) \right\} \\
&= \min\{\mu_{h(A)}(a), \mu_{h(A)}(b)\}
\end{aligned}$$

Hence h(A) is a fuzzy Z-Subalgebra of Y.

Theorem 4.2: Let $h : (X,*,0) \to (Y,*',0')$ be a Z-homomorphism of Z-algebras. If A is a fuzzy Z-Subalgebra of Y then the pre-image of A denoted by $h^{-1}(A)$ is a fuzzy Z-Subalgebra of X. Converse is true if h is an Z-epimorphism.

Proof: Let $h : (X,*,0) \to (Y,*',0')$ be a Z-homomorphism of a Z-algebra $(X,*,0)$ into a Z-algebra $(Y,*',0')$ and let A be a fuzzy Z-Subalgebra of Y.

To prove: $h^{-1}(A)$ is a fuzzy Z-Subalgebra of X.

Let $x, y \in X$. Then,

$$\begin{aligned}
\mu_{h^{-1}(A)}(x * y) &= \mu_A(h(x * y)) \\
&= \mu_A(h(x) *' h(y)) \\
&\geq \min\{\mu_A(h(x)), \mu_A(h(y))\} \\
&= \min\{\mu_{h^{-1}(A)}(x), \mu_{h^{-1}(A)}(y)\}
\end{aligned}$$

Hence $\mu_{h^{-1}(A)}(x * y) \geq \min\{\mu_{h^{-1}(A)}(x), \mu_{h^{-1}(A)}(y)\}$

Therefore, $h^{-1}(A)$ is a fuzzy Z-Subalgebra of X.

On the other hand, assume that h is an Z-epimorphism and $h^{-1}(A)$ is a fuzzy Z-Subalgebra of X.

Let y_1, $y_2 \in$ Y. Since h is an Z-epimorphism, there exists x_1, $x_2 \in X$ such that $h(x_1) = y_1$ and $h(x_2) = y_2$.

This implies $x_1=h^{-1}(y_1)$ and $x_2= h^{-1}(y_2)$.

Now, $\mu_A(y_1 *' y_2) = \mu_A(h(x_1) *' h(x_2))$

$$= \mu_A(h(x_1 * x_2))$$

$$= \mu_{h^{-1}(A)}(x_1 * x_2)$$

$$\geq \min\{\mu_{h^{-1}(A)}(x_1), \mu_{h^{-1}(A)}(x_2)\}$$

$$= \min\{\mu_A(h(x_1)), \mu_A(h(x_2))\}$$

$$= \min\{\mu_A(y_1), \mu_A(y_2)\}$$

Hence A is a fuzzy Z-Subalgebra of Y.

Definition 4.3: Let h be an Z-endomorphism of Z-algebras and A be a fuzzy set in X. We define a new fuzzy set A^h in X as $\mu_{A^h}(x) = \mu_A(h(x))$ for all $x \in X$.

Theorem 4.4: Let h be an Z-endomorphism of Z-algebra $(X,*,0)$. If A be a fuzzy Z-Subalgebra of X. Then A^h is also a fuzzy Z-Subalgebra of X.

Proof: Let h be an Z-endomorphism of Z-algebra $(X,*,0)$. Let A be a fuzzy Z-Subalgebra of X.

To prove: A^h is also a fuzzy Z-Subalgebra of X.

Let x, $y \in X$. Then

$\mu_{A^h}(x * y) = \mu_A(h(x * y))$

$$= \mu_A(h(x) * h(y))$$

$$\geq \min\{\mu_A(h(x)), \mu_A(h(y))\}$$

$$\Rightarrow \quad \mu_{A^h}(x * y) \geq \min\{\mu_{A^h}(x), \mu_{A^h}(y)\}$$

Hence A^h is a fuzzy Z-Subalgebra of X.

5. Cartesian Product of Fuzzy Z-Subalgebras of Z-algebras

In this section, we discuss the concept of Cartesian product of fuzzy Z-Subalgebras in Z-algebras.

Theorem 5.1: If A and B be fuzzy Z-subalgebras of a Z-algebra X then $A \times B$ is also a fuzzy Z-Subalgebra of $X \times X$.

Proof: Let A and B be fuzzy Z-subalgebras of a Z-algebra X.

To prove: $A \times B$ is also a fuzzy Z-Subalgebra of $X \times X$.

For any (x_1,x_2), $(y_1,y_2) \in X \times X$, we have

$$\begin{aligned}
\mu_{A\times B}((x_1,x_2)*(y_1,y_2)) &= \mu_{A\times B}(x_1*y_1,\ x_2*y_2) \\
&= \min\{\mu_A(x_1*y_1),\ \mu_B(x_2*y_2)\} \\
&\geq \min\{\min\{\mu_A(x_1),\ \mu_A(y_1)\},\ \min\{\mu_B(x_2),\ \mu_B(y_2)\}\} \\
&= \min\{\min\{\mu_A(x_1),\ \mu_B(x_2)\},\ \min\{\mu_A(y_1),\ \mu_B(y_2)\}\} \\
&= \min\{\mu_{A\times B}(x_1,x_2),\ \mu_{A\times B}(y_1,y_2)\}
\end{aligned}$$

Hence $A \times B$ is also a fuzzy Z-Subalgebra of $X \times X$.

We can generalize the above theorem as follows.

Theorem 5.2: Let $\{X_i \mid i = 1,2,\ldots,n\}$ be a finite collection of Z-algebras and $X = \prod_{i=1}^{n} X_i$. Let A_i, $i = 1,2,\ldots,n$ be fuzzy Z-Subalgebras of X_i respectively. Then $A = \prod_{i=1}^{n} A_i$ is also a fuzzy Z-Subalgebra of X.

Theorem 5.3: If B is a fuzzy Z-subalgebra of a Z-algebra X then the strongest fuzzy relation A_B is a fuzzy Z-Subalgebra of $X \times X$.

Proof: Let B be a fuzzy Z-Subalgebra of a Z-algebra X .Then for all $(x_1,y_1),(x_2,y_2) \in X \times X$,

Then $\mu_{A_B}((x_1,y_1)*(x_2,y_2)) = \mu_{A_B}(x_1*x_2, y_1*y_2)$

$$\begin{aligned}
&= \min\{\mu_B(x_1*x_2), \mu_B(y_1*y_2)\} \\
&\geq \min\{\min\{\mu_B(x_1), \mu_B(x_2)\}, \min\{\mu_B(y_1), \mu_B(y_2)\}\} \\
&= \min\{\min\{\mu_B(x_1), \mu_B(y_1)\}, \min\{\mu_B(x_2), \mu_B(y_2)\}\} \\
&= \min\{\mu_{A_B}(x_1,y_1), \mu_{A_B}(x_2,y_2)\}
\end{aligned}$$

Therefore A_B is a fuzzy Z-subalgebra of $X \times X$.

CONCLUSION

In this article, we have introduced fuzzy Z-Subalgebras in Z-algebras and discussed their properties. In future, we will study fuzzy ideals on Z-algebras and related results.

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