wjert, 2020, Vol. 6, Issue 3, 571-589. *Review Article* ISSN 2454-695X

World Journal of Engineering Research and Technology

WJERT

www.wjert.org

SJIF Impact Factor: 5.924

FRACTIONAL HARDY-TYPE INEQUALITIES

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Article Received on 08/04/2020 **Article Revised on 28/04/2020** **Article Accepted on 18/05/2020**

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ABSTRACT

Since the last few decades, the addition of abundant literature on Hardy-type inequalities proves its importance in the field (see [1], [2], [3], [4], [5], [6], [7, [8], [9], [10], [11]). We give some Hardy-type inequalities for fractional integrals and fractional derivatives in the second chapter. The main objective is to review the paper by Sajid Iqbal et. al.[12] In this article we mainly focused on the mean value theorem for Hadamard-type fractional integral and the related means for fractional integral and derivatives.

Convex Function

Definition 1.1.1

A function $\Phi : I \to R$ is called a convex function in the Jensen sense if,

$$\Phi\left(\frac{s+t}{2}\right) \le \frac{\Phi(s)+\Phi(t)}{2} \tag{1.1.1}$$

$\forall$ s, $t \in I$. A J-convex function Φ is said to be strictly J-convex sense if $\forall$ pairs of points (s,t), $s \ne t$ strict inequality holds in (1.1.1).

Theorem 1.1.2.

Let $\Phi : I \to R$ be a function on the interval $I \subseteq \mathbb{R}$ such that Φ'' exists on I. Then Φ is convex iff $\Phi''(y) \ge 0$ And if $\Phi''(y) > 0$ on I, then Φ is strictly convex on the interval.

Lemma 1.1.3.

Let $f \in C^2(I)$ and let I be a compact interval such that,

$$m \le f''(y) \le M, \forall y \in I .$$

Consider two functions Φ_1, Φ_2, defined as

$$\Phi_1(y) = \frac{My^2}{2} - f(y),$$
$$\Phi_2(y) = f(y) - \frac{my^2}{2}$$

Then Φ_1 and Φ_2 are convex on I.

Jensen's Inequality

Jensen's inequality was named after the Danish mathematician Johan Jensen who proved it in 1906.[14] In the discrete form, Jensen's inequality asserts that for a function Φ

$$\Phi\left(\frac{1}{S_n}\sum_{i=1}^{n} s_i t_i\right) \le \frac{1}{S_n}\sum_{i=1}^{n} s_i \Phi(t_i) \tag{1.2.1}$$

holds, where Φ is convex function on the interval $I \subseteq \mathrm{R}$, where s_i are positive real numbers and $t_i \in I(i = 1,2,...,n)$, while $S_n = \sum_{i=1}^{n} s_i$.

The inequality (1.2.1) is strict if Φ is strictly convex except when $t_1 = t_2 = ... = t_n$.

Theorem 1.2.2

Let (Ω, A, μ) be a probability space $-\infty \le a < b \le \infty$ and if $f, \Phi \circ f \in L^1(\mu)$ be such that $a \le f(x) \le b$ for all $t \in \Omega$. Then the inequality

$$\Phi\left(\int_\Omega f(x) d\mu(x)\right) \le \int_\Omega \Phi(f(x) d\mu(x)) \tag{1.2.2}$$

holds for any convex function $\Phi : [a,b] \to \mathbb{R}$.

AM-GM Inequality

If $a_1, a_2, \ldots .. a_n$ are non-negative and real numbers and $\lambda_1, \lambda_2 \ldots .. \lambda_n$ are non-negative and real numbers such that $\lambda_1 + \lambda_2 + \ldots .. \lambda_n = 1$.[13]

$$\prod_{k=1}^{n} a_k^{\lambda k} + \prod_{k=1}^{n} b_k^{\lambda k} \le \prod_{k=1}^{n} (a_k + b_k)^{\lambda k}$$

Weighted AM-GM inequality

$$\prod_{k=1}^{n} \left(\frac{ak}{ak+bk}\right)^{\lambda_k} \le \sum_{k=1}^{n} \lambda_k \left(\frac{ak}{ak+bk}\right)$$

Similarly

$$\prod_{k=1}^{n}\left(\frac{ak}{ak+bk}\right)^{\lambda_k} \leq \sum_{k=1}^{n} \lambda_k \left(\frac{ak}{ak+bk}\right)$$

Summing up

$$\prod_{k=1}^{n}\left(\frac{1}{(ak+bk)^{\lambda_k}}\right)\left[\prod_{k=1}^{n} a_k^{\lambda_k} + \prod_{k=1}^{n} b_k^{\lambda_k}\right] \leq \sum_{k=1}^{n} \lambda_k = 1$$

If a, b, c are the lengths of sides of a triangle and

$$2s = a + b + c$$

$$\frac{a^n}{b+c} + \frac{b^n}{c+a} + \frac{c^n}{a+b} \geq \left(\frac{2}{3}\right)^{n-2} S^{n-1}, n \geq 1$$

when $n = 1$, result equal to Nesbitt Inequality for $n \geq 2$

$$\frac{a^n}{b+c} + \frac{b^n}{c+a} + \frac{c^n}{a+b} \geq \frac{(a+b+c)^n}{3^{n-1-1}(b+c+c+a+a+b}$$

$$= \left(\frac{2}{3}\right)^{n-2} S^{n-1}$$

Fractional Integrals and Fractional Derivatives

Let $0 < m < n \leq \infty$. We denote the space of all functions on $[m,n]$ having continuous derivatives up to m order by $C^m([m,n])$, where $AC([m,n])$ is the space of all absolutely continuous function on the interval $[m,n]$. $AC^m([m,n])$ is the space of all functions $g \in C^{l-1}([m,n])$ where $g^{(l-1)} \in AC([m,n])$. For $\alpha \in \mathrm{R}$, $[\alpha]$ is the integral part of α (the integer park of k such that $k \leq \alpha < k+1$). Also, $[\alpha]$ is the ceiling of $\alpha(\min j \in \mathrm{N}, N \geq \alpha)$. We denote the space of all integrable functions on the interval (m,n) by $L_1(m,n)$ and the set of all functions that are measurable and essentially bounded on the interval (m,n) is denoted by $L_\infty(m,n)$. Evidently, $L_\infty(m,n) \subset L_1(m,n)$.

The following definition is given in [10, p.110].[15]

Definition 1.3.1.

Let $[m,n]$ be finite or infinite of $\mathbb{R}_+$ and $\alpha > 0$. The left and right-sided Hadamard-type fractional integrals of order $\alpha > 0$ are given by

$$J_{m+}^{\alpha} y(s) = \frac{1}{\Gamma}\int_m^s (\log\frac{s}{t})^{\alpha-1}\frac{y(t)dt}{t}, \text{ s} > \text{m} \qquad (1.3.1)$$

and

$$J_{n-}^{\alpha} y(s) = \frac{1}{\Gamma}\int_{s}^{n} (\log\frac{t}{s})^{\alpha-1} \frac{y(t)dt}{t}, \ \mathrm{s} < \mathrm{n} \tag{1.3.2}$$

respectively.

Definition 1.3.3

[5] Let $AC([a,b])$ be the space of all absolutely continuous functions on $[a,b]$. We denote the space of all functions $f \in C^{n}([a,b])$ with $f^{(n-1)} \in AC([a,b])$ by $AC^{n}([a,b])$. Let $\alpha \in \mathbb{R}^{+}$ & $f \in AC^{n}([a,b])$, then the Caputo fractional derivative of order α for a function $f(t)$ is defined as

$$D_{*a}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)}\int_{a}^{y} \frac{f^{n}(x)}{(y-x)^{\alpha-n+1}} dx \tag{1.3.3}$$

where, $n = [\alpha]+1$; the notation of $[\alpha]$ stands for the largest integer not greater than $[\alpha]$.

Lemma 1.3.4

[5] Let $\beta > \alpha \geq 0$. $f \in L_{1}(m,n)$ has an L_{∞} fractional derivative D_{α}^{β} in the interval $[m,n]$ & $D_{m}^{\beta-k} f(m) = 0$, k=1,....,$[\beta]+1$.

Then,

$$D_{m}^{\alpha} f(x) = \frac{1}{\Gamma(\beta-\alpha)}\int_{m}^{x} (x-y)^{\beta-\alpha-1} D_{\alpha}^{\beta} f(y)dy$$

$\forall \ \mathrm{m} \leq x \leq n$

Clearly,

$D_{m}^{\alpha} f \ \textit{is in} \ AC([m,n]) \ \textit{for} \ \beta-\alpha \geq 1,$

$D_{m}^{\alpha} f \ \textit{is in} \ C([m,n]) \ \textit{for} \ \beta-\alpha \in (0,1).$

Hence,

$D_{m}^{\alpha} f \in L_{\infty}(m,n),$

$D_{m}^{\alpha} f \in L_{1}(m,n).$

Mean Value Theorem for Fractional Integrals

Here we give the Mean Value Theorem for Riemann-Liouville Fractional Integrals and Hadamard-type Fractional Integrals.

2.1.1. Mean Value Theorem for Riemann-Liouville Fractional Integrals

Theorem 2.1.2.

Let $f,g \in C^2(I)$ where I is a compact interval, $u_i \in C([a,b])(i=1,2)$, and $r(x) \geq 0\ \forall\ x \in [a,b]$. Also let, $\frac{I_a^\alpha u(x)}{I_a^\alpha u(x)}, \frac{u(x)}{u(x)} \in I$, let $\frac{u(x)}{u(x)}$ be a non-constant. Let $Q_I(t)$ be given in (2.1.2), and $u_1(x), u_2(x)$ have Riemann-Liouville fractional integral of order $\alpha > 0$.

Then $\exists\ \zeta \in I$ such that

$$\frac{\int_a^b Q_I(x) f(\frac{u(x)}{u(x)})dx - \int_a^b r(x) f(\frac{I_a^\alpha u(x)}{I_a^\alpha u(x)})dx}{\int_a^b Q_I(x) g(\frac{u(x)}{u(x)})dx - \int_a^b r(x) g(\frac{I_a^\alpha u(x)}{I_a^\alpha u(x)})dx} = \frac{f''(\zeta)}{g''(\zeta)} \qquad (2.1.2)$$

Provided that the denominators are not equal to zero.

Proof. Let us take a function $h \in C^2(I)$ defined as
$h(x) = c_1 f(x) - c_2 g(x)$ (2.1.3)

where

$$c_1 = \int_a^b Q_I(x) g(\frac{u_1(x)}{u_2(x)})dx - \int_a^b r(x) g(\frac{I_a^\alpha u_1(x)}{I_a^\alpha u_2(x)})dx$$

$$c_2 = \int_a^b Q_I(x) f(\frac{u_1(x)}{u_2(x)})dx - \int_a^b r(x) f(\frac{I_a^\alpha u_1(x)}{I_a^\alpha u_2(x)})dx$$

By Theorem 3.1.1 with $h = f$, we have

$$c_1\int_a^b Q_I(x) f(\frac{u_1(x)}{u_2(x)})dx - c_1\int_a^b r(x) f(\frac{I_a^\alpha u_2(x)}{I_a^\alpha u_2(x)})dx - c_2\int_a^b Q_I(x) g(\frac{u_1(x)}{u_2(x)})dx + c_2\int_a^b r(x) g(\frac{I_a^\alpha u_2(x)}{I_a^\alpha u_2(x)})dx$$
$$= \frac{c_1 f''(\zeta)}{2}\int_a^b Q_I(x)(\frac{u_1(x)}{u_2(x)})^2 dx - \frac{c_1 f''(\zeta)}{2}\int_a^b r(x) f(\frac{I_a^\alpha u_2(x)}{I_a^\alpha u_2(x)})^2 dx - \frac{c_2 g''(\zeta)}{2}\int_a^b Q_I(x)(\frac{u_1(x)}{u_2(x)})^2 dx + \frac{c_2 g''(\zeta)}{2}\int_a^b r(x) f(\frac{I_a^\alpha u_2(x)}{I_a^\alpha u_2(x)})^2 dx$$

Putting the values of c_1 and c_2 in above equation, we get

$$\Rightarrow 0 = \frac{c_1 f''(\zeta)}{2}\int_a^b Q_I(x)(\frac{u_1(x)}{u_2(x)})^2 dx - \frac{c_1 f''(\zeta)}{2}\int_a^b r(x)(\frac{I_a^\alpha u_2(x)}{I_a^\alpha u_2(x)})^2 dx - \frac{c_2 g''(\zeta)}{2}\int_a^b Q_I(x)(\frac{u_1(x)}{u_2(x)})^2 dx + \frac{c_2 g''(\zeta)}{2}\int_a^b r(x)(\frac{I_a^\alpha u_2(x)}{I_a^\alpha u_2(x)})^2 dx$$
$$\Rightarrow 0 = (\frac{c_1 f''(\zeta)}{2} - \frac{c_2 g''(\zeta)}{2})(\int_a^b Q_I(x)(\frac{u_1(x)}{u_2(x)})^2 dx - \int_a^b r(x)(\frac{I_a^\alpha u_2(x)}{I_a^\alpha u_2(x)})^2 dx)$$

since

$$\int_a^b Q_I(x)(\frac{u_1(x)}{u_2(x)})^2 dx - \int_a^b r(x) f(\frac{I_a^\alpha u_2(x)}{I_a^\alpha u_2(x)})^2 dx \neq 0,$$

so we get,

$$\frac{c_1 f''(\zeta)}{2} - \frac{c_2 g''(\zeta)}{2} = 0$$

$$\Rightarrow c_1 f''(\zeta) - c_2 g''(\zeta) = 0$$

$$\Rightarrow \frac{c_2}{c_1} = \frac{f''(\zeta)}{g''(\zeta)}$$

$$\Rightarrow \frac{\int_a^b Q_I(x) f(\frac{u(x)}{u(x)})dx - \int_a^b r(x) f(\frac{I_a^\alpha u(x)}{I_a^\alpha u(x)})dx}{\int_a^b Q_I(x) g(\frac{u(x)}{u(x)})dx - \int_a^b r(x) g(\frac{I_a^\alpha u(x)}{I_a^\alpha u(x)})dx} = \frac{f''(\zeta)}{g''(\zeta)} \qquad (2.1.4)$$

This completes the proof.

Theorem 3.1.1.

Let $I \subseteq \mathbb{R}^+$ where I is a compact interval, $r(x) \geq 0 \ \forall \ x \in [a,b]$ and $\frac{I_a^\alpha v(x)}{I_a^\alpha v(x)}, \frac{v(x)}{v(x)} \in I$ where $\frac{v(x)}{v(x)}$ is non-constant and let $Q_I(t)$ be given in (2.1.2).

Then $\forall \ p,q \in \mathbb{R} \setminus \{0,1\}$ and $p \neq q$, $\exists \ \zeta$ such that

$$\left(\frac{\text{p(p-1)}\int_a^b Q_I(x)(\frac{v(x)}{v(x)})^q dx - \int_a^b r(x)(\frac{I_a^\alpha v(x)}{I_a^\alpha v(x)})^q dx}{\text{q(q-1)}\int_a^b Q_I(x)(\frac{v(x)}{v(x)})^p dx - \int_a^b r(x)(\frac{I_a^\alpha v(x)}{I_a^\alpha v(x)})^p dx} \right)^{\overline{q-p}} \qquad (3.1.1)$$

Proof.

We set, $f(x) = x^q$ and $g(x) = x^p$, $q \neq p$, $p,q \neq 0,1$. By the Theorem (3.1.2) we have

$$\frac{\int_a^b Q_I(x)(\frac{v(x)}{v(x)})^q dx - \int_a^b r(x)(\frac{I_a^\alpha v(x)}{I_a^\alpha v(x)})^q dx}{\int_a^b Q_I(x)(\frac{v(x)}{v(x)})^p dx - \int_a^b r(x)(\frac{I_a^\alpha v(x)}{I_a^\alpha v(x)})^p dx} = \frac{q(q-1)\zeta^{q-2}}{p(p-1)\zeta^{p-2}}.$$

$$\Rightarrow \zeta^{q-p} = \frac{p(p-1)\int_a^b Q_I(x)(\frac{v(x)}{v(x)})^q dx - \int_a^b r(x)(\frac{I_a^\alpha v(x)}{I_a^\alpha v(x)})^q dx}{q(q-1)\int_a^b Q_I(x)(\frac{v(x)}{v(x)})^p dx - \int_a^b r(x)(\frac{I_a^\alpha v(x)}{I_a^\alpha v(x)})^p dx}.$$

Or,

$$\Rightarrow \zeta = \left(\frac{p(p-1)\int_a^b Q_I(x)(\frac{v(x)}{v(x)})^q dx - \int_a^b r(x)(\frac{I_a^\alpha v(x)}{I_a^\alpha v(x)})^q dx}{q(q-1)\int_a^b Q_I(x)(\frac{v(x)}{v(x)})^p dx - \int_a^b r(x)(\frac{I_a^\alpha v(x)}{I_a^\alpha v(x)})^p dx} \right)^{\frac{1}{q-p}}.$$

Hence the proof is complete.

3.1.2. Mean Value Theorem for the Hadamard-type Fractional Integrals

Theorem 3.1.3.

Let, $f, g \in C^2(I)$, and I be a compact interval, $f_i \in C([a,b])(i=1,2)$ and $r(s) \geq 0\ \forall\ s \in [a,b]$. Also let $\frac{f(t)}{f(t)}, \frac{J_a^\delta f(s)}{J_a^\delta f(s)} \in I$, let $\frac{f(t)}{f(t)}$ be nonconstant, let $Z(t)$ be given in (2.1.2), and $f_1(t), f_2(t)$ have Hadamard-type fractional integral of order $\alpha > 0$. Then there exist $\exists\ \zeta \in I$ such that

$$\frac{\int_a^b Z(t) f\left\{\frac{f(t)}{f(t)}\right\} ds - \int_a^b r(s) f\left\{\frac{J_a^\delta f(s)}{J_a^\delta f(s)}\right\} ds}{\int_a^b Z(t) g\left\{\frac{f(t)}{f(t)}\right\} ds - \int_a^b r(s) g\left\{\frac{J_a^\delta f(s)}{J_a^\delta f(s)}\right\} ds} = \frac{f''(\zeta)}{g''(\zeta)} \tag{3.1.3}$$

provided that denominators are not equal to zero.

Proof.

Let us take a function $h \in C^2(I)$ defined as

$$h(s) = c_1 f(s) - c_2 g(s) \tag{3.1.4}$$

where

$$c_1 = \int_a^b Z(t) g\left\{\frac{f_1(t)}{f_2(t)}\right\} ds - \int_a^b r(s) g\left\{\frac{J_{a+}^\delta f_1(s)}{J_{a+}^\delta f_2(s)}\right\} ds,$$

$$c_2 = \int_a^b Z(t) f\left\{\frac{f_1(t)}{f_2(t)}\right\} ds - \int_a^b r(s) f\left\{\frac{J_{a+}^\delta f_1(s)}{J_{a+}^\delta f_2(s)}\right\} ds,$$

By Theorem 3.1.4 with $f = h$, we get

$$c_1 \int_a^b Z(t) f\left\{\frac{f_1(t)}{f_2(t)}\right\} ds - c_1 \int_a^b r(s) f\left\{\frac{J_{a+}^\delta f_1(s)}{J_{a+}^\delta f_2(s)}\right\} ds - c_2 \int_a^b Z(t) g\left\{\frac{f_1(t)}{f_2(t)}\right\} ds + c_2 \int_a^b r(s) g\left\{\frac{J_{a+}^\delta f_1(s)}{J_{a+}^\delta f_2(s)}\right\} ds$$
$$= \frac{c_1 f''(\zeta)}{2} \int_a^b Z(t) \left\{\frac{f_1(t)}{f_2(t)}\right\}^2 ds - \frac{c_1 f''(\zeta)}{2} \int_a^b r(s) \left\{\frac{J_{a+}^\delta f_1(s)}{J_{a+}^\delta f_2(s)}\right\}^2 ds - \frac{c_2 g''(\zeta)}{2} \int_a^b Z(t) \left\{\frac{f_1(t)}{f_2(t)}\right\}^2 ds + \frac{c_2 g''(\zeta)}{2} \int_a^b r(s) \left\{\frac{J_{a+}^\delta f_1(s)}{J_{a+}^\delta f_2(s)}\right\}^2 ds$$

Putting the values of c_1 and c_2 in above equation, we get

$$0=\left\{\frac{c_1 f''(\zeta)}{2}-\frac{c_2 g''(\zeta)}{2}\right\}\left[\int_a^b Z(t)\left\{\frac{f_1(t)}{f_2(t)}\right\}^2 ds-\int_a^b r(s)\left\{\frac{J_{a+}^{\delta} f_1(s)}{J_{a+}^{\delta} f_2(s)}\right\}^2 ds\right].$$

Since

$$\int_a^b Z(t)\left\{\frac{f_1(t)}{f_2(t)}\right\}^2 ds-\int_a^b r(s)\left\{\frac{J_{a+}^{\delta} f_1(s)}{J_{a+}^{\delta} f_2(s)}\right\}^2 ds\neq 0\ ,$$

So, we have

$$\frac{c_1 f''(\zeta)}{2}-\frac{c_2 g''(\zeta)}{2}=0$$

$$c_1 f''(\zeta)-c_2 g''(\zeta)=0$$

$$\frac{c_2}{c_1}=\frac{f''(\zeta)}{g''(\zeta)}$$

$$\Rightarrow\frac{\int_a^b Z(t) f\left\{\frac{f(t)}{f(t)}\right\}ds-f\int_a^b r(s)\left\{\frac{J_a^{\delta} f(s)}{J_a^{\delta} f(s)}\right\}ds}{\int_a^b Z(t) g\left\{\frac{f(t)}{f(t)}\right\}ds-g\int_a^b r(s)\left\{\frac{J_a^{\delta} f(s)}{J_a^{\delta} f(s)}\right\}ds}=\frac{f''(\zeta)}{g''(\zeta)} \qquad (3.1.5)$$

Hence the proof is complete.

Theorem 3.1.6

Let $I\subseteq\mathbb{R}^{+}$ and I be the compact interval where $r(s)\geq 0\ \forall\ s\in[a,b]$. Let $\frac{f(t)}{f(t)},\frac{J_a^{\delta} f(s)}{J_a^{\delta} f(s)}\in I$ and $\frac{f(t)}{f(t)}$ be the non-constant. And let $Z(y)$ be given in(2.1.4). Then for $p,q\in\mathbb{R}\setminus\{0,1\}$ and $p\neq q,\ \exists\ \zeta$ such that

$$\zeta=\left[\frac{p(p-1)\int_a^b Z(t)\left\{\frac{f(t)}{f(t)}\right\}^q ds-\int_a^b r(s)\left\{\frac{J_{a+}^{\delta} f(s)}{J_{a+}^{\delta} f(s)}\right\}^q ds}{q(q-1)\int_a^b Z(t)\left\{\frac{f(t)}{f(t)}\right\}^p ds-\int_a^b r(s)\left\{\frac{J_{a+}^{\delta} f(s)}{J_{a+}^{\delta} f(s)}\right\}^p ds}\right]^{\overline{q-p}} \qquad (3.1.7)$$

Proof.

We set, $f(s)=s^q$ and $g(s)=s^p$ where, $p\neq q\quad p,q\neq 0,1$. By the theorem (3.1.5) we have

$$\left[\frac{\int_a^b Z(t)\left\{\frac{f(t)}{f(t)}\right\}^q ds-\int_a^b r(s)\left\{\frac{J_{a+}^{\delta} f(s)}{J_{a+}^{\delta} f(s)}\right\}^q ds}{\int_a^b Z(t)\left\{\frac{f(t)}{f(t)}\right\}^p ds-\int_a^b r(s)\left\{\frac{J_{a+}^{\delta} f(s)}{J_{a+}^{\delta} f(s)}\right\}^p ds}\right]=\frac{q(q-1)\zeta^{q-2}}{p(p-1)\zeta^{p-2}}$$

$$\Rightarrow \zeta^{q-p}=\left[\frac{p(p-1)\int_a^b Z(t)\left\{\frac{f(t)}{f(t)}\right\}^q ds-\int_a^b r(s)\left\{\frac{J_{a+}^{\delta} f(s)}{J_{a+}^{\delta} f(s)}\right\}^q ds}{q(q-1)\int_a^b Z(t)\left\{\frac{f(t)}{f(t)}\right\}^p ds-\int_a^b r(s)\left\{\frac{J_{a+}^{\delta} f(s)}{J_{a+}^{\delta} f(s)}\right\}^p ds}\right]$$

Or,

$$\zeta=\left[\frac{p(p-1)\int_a^b Z(t)\left\{\frac{f(t)}{f(t)}\right\}^q ds-\int_a^b r(s)\left\{\frac{J_{a+}^{\delta} f(s)}{J_{a+}^{\delta} f(s)}\right\}^q ds}{q(q-1)\int_a^b Z(t)\left\{\frac{f(t)}{f(t)}\right\}^p ds-\int_a^b r(s)\left\{\frac{J_{a+}^{\delta} f(s)}{J_{a+}^{\delta} f(s)}\right\}^p ds}\right]^{\overline{q-p}}.$$

This completes the proof.

3.2. Mean Value Theorems for Differential Fractional Derivatives

We give the Mean Value Theorems for the Caputo Fractional Derivatives and L_∞ Caputo Fractional Derivatives in this section.

3.2.1 Mean Value Theorems for the Caputo Fractional Derivatives

Theorem 3.2.2

Let $f,g\in C^2(I)$, and I be a compact interval, $w_i \in AC^n([a,b])(i=1,2)$, and $r(x)\geq 0\ \forall\ x\in[a,b]$. Also let $\frac{w^{(n)}(x)}{w^{(n)}(x)},\frac{D_{*a}^{\alpha}w(x)}{D_{*a}^{\alpha}w(x)}\in I$, let $\frac{w^{(n)}(x)}{w^{(n)}(x)}$ be the non-constant, let $Q_D(t)$, be given in (2.2.2) and $w_1(x),w_2(x)$ have the Caputo derivative of order $\alpha>0$. Then $\exists\ \zeta\in I$ such that

$$\frac{\int_a^b Q_D(x) f\left\{\frac{w^{(n)}(x)}{w^{(n)}(x)}\right\}dx-\int_a^b r(x) f\left\{\frac{D_{*a}^{\alpha}w(x)}{D_{*a}^{\alpha}w(x)}\right\}dx}{\int_a^b Q_D(x) g\left\{\frac{w^{(n)}(x)}{w^{(n)}(x)}\right\}dx-\int_a^b r(x) g\left\{\frac{D_{*a}^{\alpha}w(x)}{D_{*a}^{\alpha}w(x)}\right\}dx}=\frac{f''(\zeta)}{g''(\zeta)} \tag{3.2.2}$$

provided that denominators are not equal to zero.

Proof.

Let us take a function $h \in C^2(I)$ defined as

$$h(x)=c_1 f(x)-c_2 g(x) \tag{3.2.3}$$

where,

$$c_1 = \int_a^b Q_D(x) g\left\{\frac{w_1^{(n)}(x)}{w_2^{(n)}(x)}\right\}dx - \int_a^b r(x) g\left\{\frac{D_{*a}^{\alpha} w_1(x)}{D_{*a}^{\alpha} w_2(x)}\right\}dx$$

$$c_2 = \int_a^b Q_D(x) f\left\{\frac{w_1^{(n)}(x)}{w_2^{(n)}(x)}\right\}dx - \int_a^b r(x) f\left\{\frac{D_{*a}^{\alpha} w_1(x)}{D_{*a}^{\alpha} w_2(x)}\right\}dx$$

By Theorem (3.2.1) with $f = h$, we have

$$c_1\int_a^b Q_D(x) f\left\{\frac{w_1^{(n)}(x)}{w_2^{(n)}(x)}\right\}dx - c_1\int_a^b r(x) f\left\{\frac{D_{*a}^{\alpha} w_1(x)}{D_{*a}^{\alpha} w_2(x)}\right\}dx - c_2\int_a^b Q_D(x) g\left\{\frac{w_1^{(n)}(x)}{w_2^{(n)}(x)}\right\}dx + c_2\int_a^b r(x) g\left\{\frac{D_{*a}^{\alpha} w_1(x)}{D_{*a}^{\alpha} w_2(x)}\right\}dx$$

$$= \frac{c_1 f''(\zeta)}{2}\int_a^b Q_D(x)\left\{\frac{w_1^{(n)}(x)}{w_2^{(n)}(x)}\right\}^2 dx - \frac{c_1 f''(\zeta)}{2}\int_a^b r(x)\left\{\frac{D_{*a}^{\alpha} w_1(x)}{D_{*a}^{\alpha} w_2(x)}\right\}^2 dx - \frac{c_2 g''(\zeta)}{2}\int_a^b Q_D(x)\left\{\frac{w_1^{(n)}(x)}{w_2^{(n)}(x)}\right\}^2 dx + \frac{c_2 g''(\zeta)}{2}\int_a^b r(x)\left\{\frac{D_{*a}^{\alpha} w_1(x)}{D_{*a}^{\alpha} w_2(x)}\right\}^2 dx$$

Putting the values of c_1 and c_2 in the above equation, we now have

$$0 = \left\{\frac{c_1 f''(\zeta)}{2} - \frac{c_2 g''(\zeta)}{2}\right\}\left[\int_a^b Q_D(x)\left\{\frac{w_1^{(n)}(x)}{w_2^{(n)}(x)}\right\}^2 dx - \int_a^b r(x)\left\{\frac{D_{*a}^{\alpha} w_1(x)}{D_{*a}^{\alpha} w_2(x)}\right\}^2 dx\right]$$

Since,

$$\int_a^b Q_D(x)\left\{\frac{w_1^{(n)}(x)}{w_2^{(n)}(x)}\right\}^2 dx - \int_a^b r(x)\left\{\frac{D_{*a}^{\alpha} w_1(x)}{D_{*a}^{\alpha} w_2(x)}\right\}^2 dx \neq 0,$$

So, we have

$$\frac{c_1 f''(\zeta)}{2} - \frac{c_2 g''(\zeta)}{2} = 0$$

$$c_1 f''(\zeta) - c_2 g''(\zeta) = 0$$

$$\frac{c_2}{c_1} = \frac{f''(\zeta)}{g''(\zeta)}$$

$$\Rightarrow \frac{\int_a^b Q_D(x) f\left\{\frac{w^{(n)}(x)}{w^{(n)}(x)}\right\}dx - \int_a^b r(x) f\left\{\frac{D_{*a}^{\alpha} w(x)}{D_{*a}^{\alpha} w(x)}\right\}dx}{\int_a^b Q_D(x) g\left\{\frac{w^{(n)}(x)}{w^{(n)}(x)}\right\}dx - \int_a^b r(x) g\left\{\frac{D_{*a}^{\alpha} w(x)}{D_{*a}^{\alpha} w(x)}\right\}dx} = \frac{f''(\zeta)}{g''(\zeta)} \tag{3.2.4}$$

This completes the proof.

Theorem 3.2.5

Let $I \subseteq \mathbb{R}^+$ and I be the compact interval, $r(x) \geq 0 \ \forall \ x \in [a,b]$. Let $\frac{D_{*a}^{\alpha} w(x)}{D_{*a}^{\alpha} w(x)}, \frac{w^{(n)}(x)}{w^{(n)}(x)} \in I$ and $\frac{w^{(n)}(x)}{w^{(n)}(x)}$ be the non-constant. Also let $Q_D(t)$ be given in (2.2.2). Then, for $p,q \in \mathbb{R} \setminus \{0,1\}$ and $p \neq q$q, $\exists \ \zeta \in I$ such that

$$\zeta = \left[\frac{p(p-1)\int_a^b Q_D(x) \left\{ \frac{w^{(n)}(x)}{w^{(n)}(x)} \right\}^q dx - \int_a^b r(x) \left\{ \frac{D_{*a}^{\alpha} w(x)}{D_{*a}^{\alpha} w(x)} \right\}^q dx}{q(q-1)\int_a^b Q_D(x) \left\{ \frac{w^{(n)}(x)}{w^{(n)}(x)} \right\}^p dx - \int_a^b r(x) \left\{ \frac{D_{*a}^{\alpha} w(x)}{D_{*a}^{\alpha} w(x)} \right\}^p dx} \right]^{\overline{q-p}} \qquad (3.2.5)$$

Proof.

We set $f(x) = x^q$ and $g(x) = x^p$, $p \neq q$, $p,q \neq 0,1$. By theorem (3.2.2) we have

$$\frac{\int_a^b Q_D(x) \left\{ \frac{w^{(n)}(x)}{w^{(n)}(x)} \right\}^q dx - \int_a^b r(x) \left\{ \frac{D_{*a}^{\alpha} w(x)}{D_{*a}^{\alpha} w(x)} \right\}^q dx}{\int_a^b Q_D(x) \left\{ \frac{w^{(n)}(x)}{w^{(n)}(x)} \right\}^p dx - \int_a^b r(x) \left\{ \frac{D_{*a}^{\alpha} w(x)}{D_{*a}^{\alpha} w(x)} \right\}^p dx} = \frac{q(q-1)\zeta^{q-2}}{p(p-1)\zeta^{p-2}},$$

$$\Rightarrow \zeta^{q-p} = \frac{p(p-1)\int_a^b Q_D(x) \left\{ \frac{w^{(n)}(x)}{w^{(n)}(x)} \right\}^q dx - \int_a^b r(x) \left\{ \frac{D_{*a}^{\alpha} w(x)}{D_{*a}^{\alpha} w(x)} \right\}^q dx}{q(q-1)\int_a^b Q_D(x) \left\{ \frac{w^{(n)}(x)}{w^{(n)}(x)} \right\}^p dx - \int_a^b r(x) \left\{ \frac{D_{*a}^{\alpha} w(x)}{D_{*a}^{\alpha} w(x)} \right\}^p dx},$$

or

$$\zeta = \left[\frac{p(p-1)\int_a^b Q_D(x) \left\{ \frac{w^{(n)}(x)}{w^{(n)}(x)} \right\}^q dx - \int_a^b r(x) \left\{ \frac{D_{*a}^{\alpha} w(x)}{D_{*a}^{\alpha} w(x)} \right\}^q dx}{q(q-1)\int_a^b Q_D(x) \left\{ \frac{w^{(n)}(x)}{w^{(n)}(x)} \right\}^p dx - \int_a^b r(x) \left\{ \frac{D_{*a}^{\alpha} w(x)}{D_{*a}^{\alpha} w(x)} \right\}^p dx} \right]^{\overline{q-p}}.$$

Hence the proof is complete.

Mean Value Theorems for L_∞ Fractional Derivative

Theorem 3.2.6

Let $\beta > \alpha \geq 0$, $f,g \in C^2(I)$ let I be a compact interval $u_i \in L_1(a,b)(i=1,2)$ has an L_∞ fractional derivative and $r(x) \geq 0 \ \forall \ x \in [a,b]$.Let $D_{\alpha}^{\beta-k} w_i(a) = 0$ for

$k=1,....,[\beta]+1(i=1,2)$, $\frac{D_a^\beta w(x)}{D_a^\beta w(x)},\frac{D_{*a}^\alpha w(x)}{D_{*a}^\alpha w(x)}\in I$, let $\frac{D_a^\beta w(x)}{D_a^\beta w(x)}$ be the non-constant, and let $Q_L(t)$ be given in (2.2.5). Then $\exists\ \zeta\in I$ such that

$$\frac{\int_a^b\left[Q_L(x)f\left\{\frac{D_a^\beta w(x)}{D_a^\beta w(x)}\right\}\right]dx-\int_a^b r(x)f\left\{\frac{D_a^\alpha w(x)}{D_a^\alpha w(x)}\right\}dx}{\int_a^b\left[Q_L(x)g\left\{\frac{D_a^\beta w(x)}{D_a^\beta w(x)}\right\}\right]dx-\int_a^b r(x)g\left\{\frac{D_a^\alpha w(x)}{D_a^\alpha w(x)}\right\}dx}=\frac{f''(\zeta)}{g''(\zeta)} \qquad (3.2.6).$$

provided that denominators are not equal to zero.

Proof.

Let us take a function $h\in C^2(I)$ defined as

$$h(x)=c_1f(x)-c_2g(x) \qquad (3.2.7)$$

where

$$c_1=\int_a^b\left[Q_L(x)g\left\{\frac{D_a^\beta w_1(x)}{D_a^\beta w_2(x)}\right\}\right]dx-\int_a^b r(x)g\left\{\frac{D_a^\alpha w_1(x)}{D_a^\alpha w_2(x)}\right\}dx$$

$$c_2=\int_a^b\left[Q_L(x)f\left\{\frac{D_a^\beta w_1(x)}{D_a^\beta w_2(x)}\right\}\right]dx-\int_a^b r(x)f\left\{\frac{D_a^\alpha w_1(x)}{D_a^\alpha w_2(x)}\right\}dx$$

By Theorem 3.2.4 with $f=h$, we have

$$c_1\int_a^b\left[Q_L(x)f\left\{\frac{D_a^\beta w_1(x)}{D_a^\beta w_2(x)}\right\}\right]dx-c_1\int_a^b r(x)f\left\{\frac{D_a^\alpha w_1(x)}{D_a^\alpha w_2(x)}\right\}dx$$
$$-c_2\int_a^b\left[Q_L(x)g\left\{\frac{D_a^\beta w_1(x)}{D_a^\beta w_2(x)}\right\}\right]dx+c_2\int_a^b r(x)g\left\{\frac{D_a^\alpha w_1(x)}{D_a^\alpha w_2(x)}\right\}dx$$
$$=\frac{c_1f''(\zeta)}{2}\int_a^b\left[Q_L(x)\left\{\frac{D_a^\beta w_1(x)}{D_a^\beta w_2(x)}\right\}^2\right]dx-\frac{c_1f''(\zeta)}{2}\int_a^b r(x)\left\{\frac{D_a^\alpha w_1(x)}{D_a^\alpha w_2(x)}\right\}^2dx$$
$$-\frac{c_2g''(\zeta)}{2}\int_a^b\left[Q_L(x)\left\{\frac{D_a^\beta w_1(x)}{D_a^\beta w_2(x)}\right\}^2\right]dx+\frac{c_2g''(\zeta)}{2}\int_a^b r(x)\left\{\frac{D_a^\alpha w_1(x)}{D_a^\alpha w_2(x)}\right\}^2dx$$

Putting the values of c_1 and c_2 in above equation, we get

$$0=\left\{\frac{c_1f''(\zeta)}{2}-\frac{c_2g''(\zeta)}{2}\right\}\left[\int_a^b Q_L(x)\left\{\frac{D_a^\beta w_1(x)}{D_a^\beta w_2(x)}\right\}^2dx-\int_a^b r(x)\left\{\frac{D_a^\alpha w_1(x)}{D_a^\alpha w_2(x)}\right\}^2dx\right]$$

Since,

$$\int_a^b Q_L(x)\left\{\frac{D_a^\beta w_1(x)}{D_a^\beta w_2(x)}\right\}^2 dx-\int_a^b r(x)\left\{\frac{D_a^\alpha w_1(x)}{D_a^\alpha w_2(x)}\right\}^2 dx\neq 0$$

So, we have

$$\frac{c_1 f''(\zeta)}{2}-\frac{c_2 g''(\zeta)}{2}=0$$

$$c_1 f''(\zeta)-c_2 g''(\zeta)=0$$

$$\frac{c_2}{c_1}=\frac{f''(\zeta)}{g''(\zeta)}$$

$$\Rightarrow\frac{\int_a^b\left[Q_L(x)\,f\left\{\frac{D_a^\beta w(x)}{D_a^\beta w(x)}\right\}\right]dx-\int_a^b r(x)\,f\left\{\frac{D_a^\alpha w(x)}{D_a^\alpha w(x)}\right\}dx}{\int_a^b\left[Q_L(x)\,g\left\{\frac{D_a^\beta w(x)}{D_a^\beta w(x)}\right\}\right]dx-\int_a^b r(x)g\left\{\frac{D_a^\alpha w(x)}{D_a^\alpha w(x)}\right\}dx}=\frac{f''(\zeta)}{g''(\zeta)} \qquad (3.2.24)$$

This completes the proof.

Theorem 3.2.8.

Let, $I\subseteq\mathbb{R}^+$, and I be a compact interval, $u_i\in U(v,k)(i=1,2)$, and $r(x)\geq 0\ \forall\ x\in[a,b]$. Let, $\frac{D_a^\beta w(x)}{D_a^\beta w(x)},\frac{D_a^\alpha w(x)}{D_a^\alpha w(x)}\in I$, $\frac{D_a^\beta w(x)}{D_a^\beta w(x)}$ be non-constant and let that $Q_L(t)$ be given in (2.2.5). Then for $p,q\in\mathbb{R}\setminus\{0,1\}$ and $p\neq q$, $\exists\ \zeta\in I$ such that,

$$\zeta=\left[\frac{p(p-1)\int_a^b\left[Q_L(x)\left\{\frac{D_a^\beta w(x)}{D_a^\beta w(x)}\right\}^q\right]dx-\int_a^b r(x)\left\{\frac{D_a^\alpha w(x)}{D_a^\alpha w(x)}\right\}^q dx}{q(q-1)\int_a^b\left[Q_L(x)\left\{\frac{D_a^\beta w(x)}{D_a^\beta w(x)}\right\}\right]^p dx-\int_a^b r(x)\left\{\frac{D_a^\alpha w(x)}{D_a^\alpha w(x)}\right\}^p dx}\right]^{\overline{q-p}} \qquad (3.2.9)$$

Proof.

We set $f(x)=x^q$ and $g(x)=x^p$, $p\neq q$, $p,q\neq 0,1$. By theorem 3.2.5 we have

$$\frac{\int_a^b Q_L(x)\left\{\frac{D_a^\beta w(x)}{D_a^\beta w(x)}\right\}^q dx-\int_a^b r(x)\left\{\frac{D_a^\alpha w(x)}{D_a^\alpha w(x)}\right\}^q dx}{\int_a^b Q_L(x)\left\{\frac{D_a^\beta w(x)}{D_a^\beta w(x)}\right\}^p dx-\int_a^b r(x)\left\{\frac{D_a^\alpha w(x)}{D_a^\alpha w(x)}\right\}^p dx}=\frac{q(q-1)\zeta^{q-2}}{p(p-1)\zeta^{p-2}} \qquad (3.2.10)$$

$$\Rightarrow \zeta^{q-p} = \frac{p(p-1)\int_a^b \left[Q_L(x) \left\{ \frac{D_a^{\beta} w(x)}{D_a^{\beta} w(x)} \right\}^q \right] dx - \int_a^b r(x) \left\{ \frac{D_a^{\alpha} w(x)}{D_a^{\alpha} w(x)} \right\}^q dx}{q(q-1)\int_a^b \left[Q_L(x) \left\{ \frac{D_a^{\beta} w(x)}{D_a^{\beta} w(x)} \right\} \right]^p dx - \int_a^b r(x) \left\{ \frac{D_a^{\alpha} w(x)}{D_a^{\alpha} w(x)} \right\}^p dx} \qquad (3.2.11)$$

Or,

$$\zeta = \left[\frac{p(p-1)\int_a^b \left[Q_L(x) \left\{ \frac{D_a^{\beta} w(x)}{D_a^{\beta} w(x)} \right\}^q \right] dx - \int_a^b r(x) \left\{ \frac{D_a^{\alpha} w(x)}{D_a^{\alpha} w(x)} \right\}^q dx}{q(q-1)\int_a^b \left[Q_L(x) \left\{ \frac{D_a^{\beta} w(x)}{D_a^{\beta} w(x)} \right\} \right]^p dx - \int_a^b r(x) \left\{ \frac{D_a^{\alpha} w(x)}{D_a^{\alpha} w(x)} \right\}^p dx} \right]^{\overline{q-p}} \qquad (3.2.12)$$

Hence the proof is complete.

Means for the Fractional Integrals

In this chapter, we give the Means for Riemann-Liouville fractional integrals and Hadamard-type fractional integrals. See[6] for detailed reference.

4.1.1 Means for the Hadamard-type Fractional Integrals

The means for the Hadamard-type fractional integral is defined as,

$$M_{p,q} = \left(\frac{\Pi_q}{\Pi_p} \right)^{\overline{q-p}} \qquad (4.1.1)$$

where $p, q \in \square$,

$$\Pi_q = p(p-1)\int_a^b Z(y) \left\{ \frac{f_1(y)}{f_2(y)} \right\}^q dx - \int_a^b r(x) \left\{ \frac{J_{a+}^{\delta} f_1(x)}{J_{a+}^{\delta} f_2(x)} \right\}^q dx$$

and

$$\Pi_p = q(q-1)\int_a^b Z(y) \left\{ \frac{f_1(y)}{f_2(y)} \right\}^p dx - \int_a^b r(x) \left\{ \frac{J_{a+}^{\delta} f_1(x)}{J_{a+}^{\delta} f_2(x)} \right\}^p dx$$

In the following, we will discuss limiting cases of above mean for $p, q \neq 0,1$

$$\lim_{q \to p} M_{p,q} = M_{p,p}$$

$$= \exp \left[\frac{\int_a^b Z(y) \left\{ \frac{f(y)^p}{f(y)^p} \right\} \log \left\{ \frac{f(y)}{f(y)} \right\} dx - \int_a^b r(x) \left\{ \frac{J_a^{\delta} f(x)^p}{J_a^{\delta} f(x)^p} \right\} \log \left\{ \frac{J_a^{\delta} f(x)}{J_a^{\delta} f(x)} \right\} dx}{\int_a^b Z(y) \left\{ \frac{f(y)^p}{f(y)^p} \right\} dx - \int_a^b r(x) \left\{ \frac{J_a^{\delta} f(x)^p}{J_a^{\delta} f(x)^p} \right\} dx} - \frac{2p-1}{p(p-1)} \right]$$

$$\lim_{p\to 0} M_{p,p} = M_{0,0}$$

$$== \exp\left[\frac{\int_a^b Z(y)\log^2\left\{\frac{f(y)}{f(y)}\right\}dx - \int_a^b r(x)\log^2\left\{\frac{J_a^\delta f(x)}{J_a^\delta f(x)}\right\}dx}{2\int_a^b Z(y)\log\left\{\frac{f(y)}{f(y)}\right\}dx - \int_a^b r(x)\log\left\{\frac{J_a^\delta f(x)}{J_a^\delta f(x)}\right\}dx} + 1\right]$$

$$\lim_{p\to 1} M_{p,p} = M_{1,1}$$

$$= \exp\left[\frac{\int_a^b Z(y)\left\{\frac{f(y)}{f(y)}\right\}\log^2\left\{\frac{f(y)}{f(y)}\right\}dx - \int_a^b r(x)\left\{\frac{J_a^\delta f(x)}{J_a^\delta f(x)}\right\}\log^2\left\{\frac{J_a^\delta f(x)}{J_a^\delta f(x)}\right\}dx}{2\int_a^b Z(y)\left\{\frac{f(y)}{f(y)}\right\}\log\left\{\frac{f(y)}{f(y)}\right\}dx - \int_a^b r(x)\left\{\frac{J_a^\delta f(x)}{J_a^\delta f(x)}\right\}\log\left\{\frac{J_a^\delta f(x)}{J_a^\delta f(x)}\right\}dx} - 1\right]$$

$$\lim_{q\to 0} M_{p,q} = M_{p,0}$$

$$= \exp\left[\frac{\int_a^b Z(y)\left\{\frac{f(y)^p}{f(y)^p}\right\}dx - \int_a^b r(x)\left\{\frac{J_a^\delta f(x)^p}{J_a^\delta f(x)^p}\right\}dx}{\left[2\int_a^b Z(y)\log\left\{\frac{f(y)}{f(y)}\right\}dx - \int_a^b r(x)\log\left\{\frac{J_a^\delta f(x)^p}{J_a^\delta f(x)^p}\right\}dx\right]p(p-1)}\right]^{\frac{1}{p}}$$

$$\lim_{q\to 1} M_{p,q} = M_{p,1}$$

$$= \exp\left[\frac{\left[\int_a^b Z(y)\left\{\frac{f(y)}{f(y)}\right\}\log\left\{\frac{f(y)}{f(y)}\right\}dx - \int_a^b r(x)\left\{\frac{J_a^\delta f(x)}{J_a^\delta f(x)}\right\}\log\left\{\frac{J_a^\delta f(x)}{J_a^\delta f(x)}\right\}dx\right]p(p-1)}{\int_a^b Z(y)\left\{\frac{f(y)^p}{f(y)^p}\right\}dx - \int_a^b r(x)\left\{\frac{J_a^\delta f(x)^p}{J_a^\delta f(x)^p}\right\}dx}\right]^{\frac{1}{-p}}$$

4.2 Means for the Fractional Derivatives

4.2.1 Means for Caputo fractional derivative

Right-hand side of (3.1.12) is the mean, then for the distinct $p,q \in \mathbb{R}$, it can be written as

$$M_{p,q} = \left(\frac{\Pi_q}{\Pi_p}\right)^{\frac{1}{q-p}} \tag{4.2.1}$$

Where,

$$\Pi_q = p(p-1)\int_a^b Q_D(x)\left\{\frac{w_1^{(n)}(x)}{w_2^{(n)}(x)}\right\}^q dx - \int_a^b r(x)\left\{\frac{D_{*a}^\alpha w_1(x)}{D_{*a}^\alpha w_2(x)}\right\}^q dx$$

and

$$\Pi_p = q(q-1)\int_a^b Q_D(x)\left\{\frac{w_1^{(n)}(x)}{w_2^{(n)}(x)}\right\}^p dx - \int_a^b r(x)\left\{\frac{D_{*a}^{\alpha} w_1(x)}{D_{*a}^{\alpha} w_2(x)}\right\}^p dx$$

Moreover, we can extend these means, so in limiting cases for $p, q \neq 0,1$

$$\lim_{q\to p} M_{p,q} = M_{p,p}$$

$$= \exp\left[\frac{\int_a^b Q_D(x)\left\{\frac{w^{(n)}(x)^p}{w^{(n)}(x)^p}\right\}\log\left\{\frac{w^{(n)}(x)}{w^{(n)}(x)}\right\}dx - \int_a^b r(x)\left\{\frac{D_{*a}^{\alpha} w(x)^p}{D_{*a}^{\alpha} w(x)^p}\right\}\log\left\{\frac{D_{*a}^{\alpha} w(x)}{D_{*a}^{\alpha} w(x)}\right\}dx}{\int_a^b Q_D(x)\left\{\frac{w^{(n)}(x)^p}{w^{(n)}(x)^p}\right\}dx - \int_a^b r(x)\left\{\frac{D_{*a}^{\alpha} w(x)^p}{D_{*a}^{\alpha} w(x)^p}\right\}dx} - \frac{2p-1}{p(p-1)}\right]$$

$$\lim_{p\to 0} M_{p,p} = M_{0,0}$$

$$== \exp\left[\frac{\int_a^b Q_D(x)\log^2\left\{\frac{w^{(n)}(x)}{w^{(n)}(x)}\right\}dx - \int_a^b r(x)\log^2\left\{\frac{D_{*a}^{\alpha} w(x)}{D_{*a}^{\alpha} w(x)}\right\}dx}{2\int_a^b Q_D(x)\log\left\{\frac{w^{(n)}(x)}{w^{(n)}(x)}\right\}dx - \int_a^b r(x)\log\left\{\frac{D_{*a}^{\alpha} w(x)}{D_{*a}^{\alpha} w(x)}\right\}dx} + 1\right]$$

$$\lim_{p\to 1} M_{p,p} = M_{1,1}$$

$$= \exp\left[\frac{\int_a^b Q_D(x)\left\{\frac{w^{(n)}(x)}{w^{(n)}(x)}\right\}\log^2\left\{\frac{w^{(n)}(x)}{w^{(n)}(x)}\right\}dx - \int_a^b r(x)\left\{\frac{D_{*a}^{\alpha} w(x)}{D_{*a}^{\alpha} w(x)}\right\}\log^2\left\{\frac{D_{*a}^{\alpha} w(x)}{D_{*a}^{\alpha} w(x)}\right\}dx}{2\int_a^b Q_D(x)\left\{\frac{w^{(n)}(x)}{w^{(n)}(x)}\right\}\log\left\{\frac{w^{(n)}(x)}{w^{(n)}(x)}\right\}dx - \int_a^b r(x)\left\{\frac{D_{*a}^{\alpha} w(x)}{D_{*a}^{\alpha} w(x)}\right\}\log\left\{\frac{D_{*a}^{\alpha} w(x)}{D_{*a}^{\alpha} w(x)}\right\}dx} - 1\right]$$

$$\lim_{q\to 0} M_{p,q} = M_{p,0}$$

$$= \exp\left[\frac{\int_a^b Q_D(x)\left\{\frac{w^{(n)}(x)^p}{w^{(n)}(x)^p}\right\}dx - \int_a^b r(x)\left\{\frac{D_{*a}^{\alpha} w(x)^p}{D_{*a}^{\alpha} w(x)^p}\right\}dx}{\left[2\int_a^b Q_D(x)\log\left\{\frac{w^{(n)}(x)}{w^{(n)}(x)}\right\}dx - \int_a^b r(x)\log\left\{\frac{D_{*a}^{\alpha} w(x)^p}{D_{*a}^{\alpha} w(x)^p}\right\}dx\right] p(p-1)}\right]^{\overline{p}}$$

$$= \exp\left[\frac{\left[\int_a^b Q_D(x)\left\{\frac{w^{(n)}(x)^p}{w^{(n)}(x)^p}\right\}\log\left\{\frac{w^{(n)}(x)}{w^{(n)}(x)}\right\}dx - \int_a^b r(x)\left\{\frac{D_{*a}^{\alpha} w(x)^p}{D_{*a}^{\alpha} w(x)^p}\right\}\log\left\{\frac{D_{*a}^{\alpha} w(x)}{D_{*a}^{\alpha} w(x)}\right\}dx\right] p(p-1)}{\int_a^b Q_D(x)\left\{\frac{w^{(n)}(x)^p}{w^{(n)}(x)^p}\right\}dx - \int_a^b r(x)\left\{\frac{D_{*a}^{\alpha} w(x)^p}{D_{*a}^{\alpha} w(x)^p}\right\}dx}\right]^{\overline{-p}}$$

4.2.2 Means for L_∞ fractional derivatives

We define means for L_∞ fractional derivative as $p, q \in$ □ where, $p, q \in$ □

$$\Pi_q = p(p-1)\int_a^b Q_L(x)\left\{\frac{D_a^{\beta} w_1(x)}{D_a^{\beta} w_2(x)}\right\}^q dx - \int_a^b r(x)\left\{\frac{D_a^{\alpha} w_1(x)}{D_a^{\alpha} w_2(x)}\right\}^q dx$$

and

$$\Pi_p = q(q-1)\int_a^b Q_L(x)\left\{\frac{D_a^\beta w_1(x)}{D_a^\beta w_2(x)}\right\}^p dx - \int_a^b r(x)\left\{\frac{D_a^\alpha w_1(x)}{D_a^\alpha w_2(x)}\right\}^p dx$$

In following we discuss limiting cases of above mean for $p, q \neq 0,1$

$$\lim_{q\to p} M_{p,q} = M_{p,p}$$

$$= \exp\left[\frac{\int_a^b Q_L(x)\left\{\frac{D_a^\beta w(x)^p}{D_a^\beta w(x)^p}\right\}\log\left\{\frac{D_a^\beta w(x)}{D_a^\beta w(x)}\right\}dx - \int_a^b r(x)\left\{\frac{D_a^\alpha w(x)^p}{D_a^\alpha w(x)^p}\right\}\log\left\{\frac{D_a^\alpha w(x)}{D_a^\alpha w(x)}\right\}dx}{\int_a^b Q_L(x)\left\{\frac{D_a^\beta w(x)^p}{D_a^\beta w(x)^p}\right\}dx - \int_a^b r(x)\left\{\frac{D_a^\alpha w(x)^p}{D_a^\alpha w(x)^p}\right\}dx} - \frac{2p-1}{p(p-1)}\right]$$

$$\lim_{p\to 0} M_{p,p} = M_{0,0}$$

$$== \exp\left[\frac{\int_a^b Q_L(x)\log^2\left\{\frac{D_a^\beta w(x)}{D_a^\beta w(x)}\right\}dx - \int_a^b r(x)\log^2\left\{\frac{D_a^\alpha w(x)}{D_a^\alpha w(x)}\right\}dx}{2\left[\int_a^b Q_L(x)\log\left\{\frac{D_a^\beta w(x)}{D_a^\beta w(x)}\right\}dx - \int_a^b r(x)\log\left\{\frac{D_a^\alpha w(x)}{D_a^\alpha w(x)}\right\}dx\right]} + 1\right]$$

$$\lim_{p\to 1} M_{p,p} = M_{1,1}$$

$$= \exp\left[\frac{\int_a^b Q_L(x)\left\{\frac{D_a^\beta w(x)}{D_a^\beta w(x)}\right\}\log^2\left\{\frac{D_a^\beta w(x)}{D_a^\beta w(x)}\right\}dx - \int_a^b r(x)\left\{\frac{D_a^\alpha w(x)}{D_a^\alpha w(x)}\right\}\log^2\left\{\frac{D_a^\alpha w(x)}{D_a^\alpha w(x)}\right\}dx}{2\int_a^b Q_L(x)\left\{\frac{D_a^\beta w(x)}{D_a^\beta w(x)}\right\}\log\left\{\frac{D_a^\beta w(x)}{D_a^\beta w(x)}\right\}dx - \int_a^b r(x)\left\{\frac{D_a^\alpha w(x)}{D_a^\alpha w(x)}\right\}\log\left\{\frac{D_a^\alpha w(x)}{D_a^\alpha w(x)}\right\}dx} - 1\right]$$

$$\lim_{q\to 0} M_{p,q} = M_{p,0}$$

$$= \exp\left[\frac{\int_a^b Q_L(x)\left\{\frac{D_a^\beta w(x)^p}{D_a^\beta w(x)^p}\right\}dx - \int_a^b r(x)\left\{\frac{D_a^\alpha w(x)^p}{D_a^\alpha w(x)^p}\right\}dx}{\left[2\int_a^b Q_L(x)\log\left\{\frac{D_a^\beta w(x)}{D_a^\beta w(x)}\right\}dx - \int_a^b r(x)\log\left\{\frac{D_a^\alpha w(x)^p}{D_a^\alpha w(x)^p}\right\}dx\right]p(p-1)}\right]^{\bar{p}}$$

$$\lim_{q\to 1} M_{p,q} = M_{p,1}$$

$$= \exp\left[\frac{\left[\int_a^b Q_L(x)\left\{\frac{D_a^\beta w(x)}{D_a^\beta w(x)}\right\}\log\left\{\frac{D_a^\beta w(x)}{D_a^\beta w(x)}\right\}dx - \int_a^b r(x)\left\{\frac{D_a^\alpha w(x)}{D_a^\alpha w(x)}\right\}\log\left\{\frac{D_a^\alpha w(x)}{D_a^\alpha w(x)}\right\}dx\right]p(p-1)}{\int_a^b Q_L(x)\left\{\frac{D_a^\beta w(x)^p}{D_a^\beta w(x)^p}\right\}dx - \int_a^b r(x)\left\{\frac{D_a^\alpha w(x)^p}{D_a^\alpha w(x)^p}\right\}dx}\right]^{\overline{-p}}$$

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