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VECTOR ANALYSIS OF ELECTROMAGNETIC FIELDS AND WAVES USING, GREEN'S THEOREM FOR POISSON'S AND LAPLACE, S EQUATIONS, STOKE'S THEOREM, MAXWELL EQUATION AND LAURENT'S FORCE LAW.

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ABSTRACT

Vector analysis, having been acquainted with gradient of scalar fields, divergence of vector fields, curl of vector fields, line integral, surface integral and volume integral of vector fields with applications of Gauss' theorem, here , the work emphasizes on Green's theoremwith different approach to Gauss divergence theorem, where for instance, if $\nabla \times F = 0$, then integral around a closed path is zero, meaning, the work done is zero and the force field is conservative. Application of Green theorem on Poisson's equation, where scalar point function vanishes outside a finite region. It also shows that, for Laplace equation $\nabla^2 \emptyset = 0$, where, $\emptyset$, which is a scalar point function for every point of the region,is said to be HARMONIC in that region. Maxwell's equation is fully analysed. Stoke's theorem and its tangential line integral of vector field over any closed surface over any closed surface S bounded by a curve and which is equivalent to normal surface integral of curl of the vector field over the surface. Also Laurent's force law analyses force field through concentric circles.

INDEXTERMS: Green's Theorem, Poisson's. Laplace's and Maxwell's Equations; Stoke's theorem.

INTRODUCTION

With earlier knowledge of gradient of a scalar field, divergence of a vector field and curl of a vector field. Green's identities and theorem are applied to explain Poisson's and Laplace's equations. Maxwell's equation (Matthew, S. 2014), is used to analytically discussed relationship between Electric field intensity E and magnetic field intensity or magnetic flux density H. Stoke's theorem deals with tangential line integral of vector field over any closed surface bounded by a curve and it is equal to the normal surface integral of curl of the vector field over the surface is zero (Matthew, S. 2014). Laurent's force law (David, H. et al, 2016) is vividly explained with its Taylor expansion in Fourier series.

Two Green Identities

First Identity, If $\emptyset$ *and* φ are scalar point functions having continuous derivatives of the second order at least, the $\iiint_v(\emptyset\nabla^2\varphi + \nabla\varphi.\nabla\emptyset)\ dv = \iint_s(\emptyset\nabla\varphi).ds$.

The divergence theorem is $\iiint_v \nabla.F dv = \iint_s F.nds$.

Taking F = $\emptyset\nabla\varphi$, $\iiint_v \nabla.(\emptyset\nabla\varphi)dv = \iint_s(\emptyset\nabla\varphi).ds$.

But, $\nabla.(\emptyset\nabla\varphi) = \emptyset(\nabla.\nabla\varphi) + (\nabla\emptyset).(\nabla\varphi)$

$=\emptyset(\nabla^2\varphi) + (\nabla\emptyset).(\nabla\varphi)$.

$\therefore \iiint_v \nabla.(\emptyset\nabla\varphi)dv = \iiint_v(\emptyset\nabla^2\varphi) + (\nabla\emptyset).(\nabla\varphi).dv = \iint_s(\emptyset\nabla\varphi).ds$.

Second Identity $\iiint_v(\emptyset\nabla^2\varphi) - (\varphi\nabla^2\emptyset)dv = \iint_s(\emptyset\nabla\varphi) - (\varphi\nabla\emptyset).ds$.

Putting F = $\varphi\nabla\emptyset$, divergence theorem gives

$\iiint_v \nabla.(\varphi\nabla\emptyset)dv = \iint_s(\varphi\nabla\emptyset).nds = \iint_s(\varphi\nabla\emptyset).ds$

But $\nabla.(\varphi\nabla\emptyset = \varphi(\nabla.\nabla\emptyset) + (\nabla\varphi).(\nabla\emptyset)$

$\varphi(\nabla^2\emptyset) + (\nabla\emptyset).(\nabla\varphi)$.

$\therefore \iiint_v \nabla.(\varphi\nabla\emptyset)\ dv = \iiint_v(\varphi(\nabla^2\emptyset) + (\nabla\emptyset).(\nabla\varphi))dv$

And $\iiint_v (\varphi(\nabla^2\emptyset) + (\nabla\emptyset).(\nabla\varphi))dv = \iint_s (\varphi\nabla\emptyset).ds.$ (1)

Green 1st Identity is $\iiint_v ((\emptyset\nabla^2\varphi) + (\nabla\emptyset).(\nabla\varphi))dv = \iint_s (\emptyset\nabla\varphi).ds.$ (2)

Subtracting (1) from (2) $\iiint_v ((\emptyset\nabla^2\varphi) - \varphi(\nabla^2\emptyset))\, dv = \iint_s ((\emptyset\nabla\varphi) - (\varphi\nabla\emptyset))\, .ds.$

To show that the 2nd Green's Identity can be written as

$\iiint_v ((\emptyset\nabla^2\varphi) - \varphi(\nabla^2\emptyset))\, dv = \iint_s ((\emptyset\frac{\partial\varphi}{\partial n}) - (\varphi\frac{\partial\emptyset}{\partial n}))\, .ds.$

The Green's Identity is

$\iiint_v ((\emptyset\nabla^2\varphi) - \varphi(\nabla^2\emptyset))\, dv = \iint_s ((\emptyset\nabla\varphi) - (\varphi\nabla\emptyset)).ds$

$= \iint_s ((\emptyset\nabla\varphi) - (\varphi\nabla\emptyset))\, .nds.$ Where n is unit normal vector (Gupta,a. 1980).

$\therefore \nabla\varphi.n = \frac{\partial\varphi}{\partial n}; \nabla\emptyset.n = \frac{\partial\emptyset}{\partial n}.$

Green Theorem in the Plane

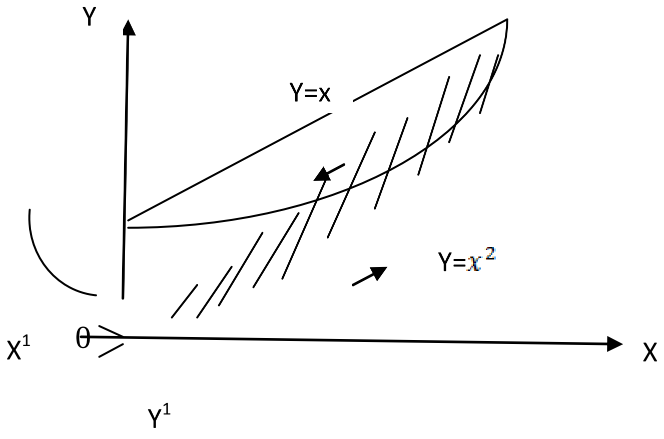
Let R be a closed region in the x-y plane bounded by a simple closed curve C and Ø and φ are two continuously differentiable functions of x and y, the Green theorem in the plane is stated as

$\oint_c (\varphi dx + \emptyset dy) \equiv \iint_R (\frac{\partial\emptyset}{\partial x} - \frac{\partial\varphi}{\partial y})dxdy,$ where C is traversed in position (anti-clockwise direction).

Vector form of Green Theorem in the plane

If f=φi + Øj and r =xi + yj, $\oint_c F.dr \equiv \iint_R (\nabla \times f).kds.$

To Verify Green Theorem in the plane for $\oint_c (xy + y^2)dx + x^2dy$, where C is the closed curve of the region bounded by y = x^2 and y = x. [1,1]



The shaded region shown in the figure represents the positive direction traversed by the closed region C, made up of a parabola and a straight line.

Given $\varphi = xy + y^2$ and $\emptyset = x^2$, Evaluating the integral along y = x^2, we have,

$\oint_c(\varphi dx + \emptyset dy) = \int_0^1 (x.x^2 + x^4)dx + 2x^3dx$ or $\int_0^1 (x^3 + x^4)dx + ydy$,

$$= \int_{x=0}^{1} (x^3 + x^4)dx + \int_{y=0}^{1} ydy$$

$$= (\frac{x}{4} + \frac{x^5}{5})_0^1 + (\frac{y^2}{2})_0^1 = \frac{19}{20}.$$

Evaluating the integral along y = x, we have, $\oint_c(\varphi dx + \emptyset dy) = \int_0^1 (x^2 + x^2)dx + ydy$ = -1.

∴ The required integral = $\frac{19}{20} - 1 = -\frac{1}{20}$.

Also $\frac{\partial\emptyset}{\partial x} = 2x \; and \; \frac{\partial\varphi}{\partial y} = x + 2y \Rightarrow \iint_R (\frac{\partial\emptyset}{\partial x} - \frac{\partial\varphi}{\partial y})$dxdy.

It is evident that $\int (\varphi dx + \emptyset dy) \equiv \iint_R (\frac{\partial\emptyset}{\partial x} - \frac{\partial\varphi}{\partial y})$dxdy = $-\frac{1}{20}$. (Green theorem verified).

Physical Interpretation of the Green Theorem

Assuming that F represents force field acting on a particle whose position vector is r, the integral $\oint_c F.$ dr may be interpreted as expressing work done in moving the particle round the closed path C and is evaluated by the value of $\nabla \times F$.

As a particular case if $\nabla \times F = 0$, then the integral around a closed path is zero. It means the work done is zero and the force field is conservative.

If the integral round a closed path is independent of the path joining any two points in the plane i.e the integral around the closed path is zero.

Then, $\nabla \times F = 0$, where F = φi + ∅j, i.e.

$(-\frac{\partial \emptyset}{\partial z} + \frac{\partial \varphi}{\partial z})$j + $(\frac{\partial \emptyset}{\partial x} - \frac{\partial \varphi}{\partial y})$k = 0. Giving $\frac{\partial \emptyset}{\partial x} = \frac{\partial \varphi}{\partial y}$.

Poisson's Equation with its Solution

Let ∅ be a scalar point function vanishing outside a finite region, then the equation,

$\nabla^2 \emptyset = -4\pi\rho$ (Poisson's Equation) (Gupta,b, 1980) (1)

Green Formula is $4\pi\emptyset(\rho) = \int_s \left(\frac{1}{r}\nabla\emptyset - \emptyset\nabla\left(\frac{1}{r}\right)\right).ds - \int_v \frac{1}{r}\nabla^2\emptyset \mathrm{dv}$ (2)

For a region bounded by a surface S, we have,

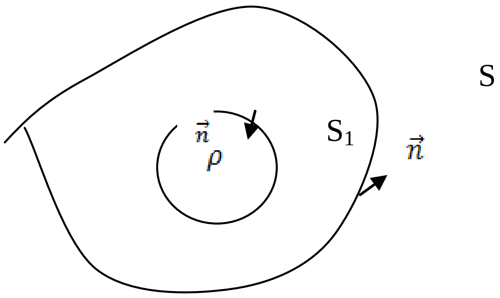
$4\pi\emptyset(\rho) = \int_s \left(\frac{1}{r}\nabla\emptyset - \emptyset\nabla\left(\frac{1}{r}\right)\right).ds + 4\pi\int_v \frac{\rho}{r}\mathrm{dv}$ by (1) (3)

In case, the region tends to infinity, S also recedes to infinity. For large value of r, ∅ is of the form $\frac{\lambda}{r}$ where λ remain bounded $|\nabla\emptyset|$ is of the form $\frac{\lambda}{r^2}$, so that $\int_s \left(\frac{1}{r}\nabla\emptyset - \emptyset\nabla\left(\frac{1}{r}\right)\right).ds \rightarrow 0$

Then, $4\pi\emptyset(\rho) = 4\pi\int \frac{\rho}{r}\mathrm{dv}$

or $\emptyset(\rho) = \int \frac{\rho}{r}\mathrm{dv}$.

The volume integral being carried over the whole space remain the same as the volume integral over bthe region outside, at which ρ is the charge density and P is a fixed point within the region which is r distance to or from a variable point of the region and r is its positive vector relative to P. Note,region is never spherical until a small sphere is drawn within the region as s_1 enclosing P with smaller radius of ε. Taking $\varphi = \frac{1}{r}$ as a scalar point function which has uniform finite and continuous derivatives up to the 2nd order in a region enclosed by a closed surface S. Clearly in the region V, bounded by S and $s_{1,}\varphi$ is twice continuously differentiable.



We know that $\nabla\varphi = \nabla(\frac{1}{r}) = \frac{1}{(x^2+y^2+z^2)^{\frac{1}{2}}} = \frac{r}{r^2}$

And $\nabla^2\varphi = \nabla(-\frac{r}{r^2}) = 0.$

Applying Green's identity to the region bounded by S and$s_{1,}$ we get,

$$\int_v(-\frac{1}{r}\nabla^2\emptyset + 0)dv = \int_s\left(\emptyset\frac{\partial}{\partial n}\frac{1}{r} - \left(\frac{1}{r}\right)\frac{\partial\emptyset}{\partial n}\right).ds + \int_{s_{1,}}\left(\emptyset\frac{\partial}{\partial n}\frac{1}{r} - \left(\frac{1}{r}\right)\frac{\partial\emptyset}{\partial n}\right).ds_{1,} \quad (1)$$

At $s_{1,}$ surface a unit normal drawn outward from the region considered is towards p so that

$\frac{\partial}{\partial n}\frac{1}{r}$ on $s_{1,}$ $=[\nabla\left(\frac{1}{r}\right).n]_{r=\varepsilon} \Rightarrow [\left(-\frac{r}{r^3}\right)(-\frac{r}{r})]_{r=\varepsilon}$

Vector equivalent;

If, f = $f_1 i + f_{2j} + f_3 k$, F = $F_1 i + F_{2j} + F_3 k$, where, ∇^2F = - 4πf, then F(P) = $\int \frac{f}{r}$dv.

∇^2F = - 4πf, is equivalent to

$\nabla^2 F_1$ = - 4πf_1,

$\nabla^2 F_2$ = - 4πf_2,

$\nabla^2 F_3$ = - 4πf_3,

Thus for a point P of the region, $F_1(P) = \int \frac{f_1}{r}$dv,

$$F_2(P) = \int \frac{f_2}{r}\text{dv},$$

$$F_3(P) = \int \frac{f_3}{r}\text{dv},$$

Imposing suitable conditions to f_1, f_2, f_3 and multiplying these relations by I, j, k respectively, and then adding, we have

$iF_1(P) + jF_2(P) + kF_3(P) = \int (if_1 + jf_2 + kf_3)\ \frac{1}{r}$dv,

$F(P) = \int \frac{f}{r}$dv,

Laplace's Equation with its Equations

If, for a twice differentiable scalar point function $\emptyset$, $\nabla^2 \emptyset = 0$, is true for every point of the region, the function $\emptyset$ is said to be HARMONIC in the region (Gupta,c, 1980).

The equation $\nabla^2 \emptyset = 0$ (Laplace's Equation) (1)

Green's Formular $4\pi\emptyset(\rho) = \int_s \left(\frac{1}{r}\nabla\emptyset - \emptyset\nabla\left(\frac{1}{r}\right)\right).ds - \int_v \frac{1}{r}\nabla^2\emptyset$dv

$$= \int_s \left(\frac{1}{r}\nabla\emptyset - \emptyset\nabla\left(\frac{1}{r}\right)\right).ds \text{ by (1)}$$

Which follows that the harmonic function $\emptyset$ at any point within the region can be expressed in terms of the value of $\emptyset$ and $\frac{\partial \emptyset}{\partial n}$ at any point of the surface enclosing the region.

Maxwell's Equations

If $\nabla.\mathrm{E} = 0$, $\nabla.\mathrm{H} = 0$, $\nabla \times \mathrm{E} = -\frac{\partial \mathrm{H}}{\partial \mathrm{t},}$, $\nabla \times \mathrm{H} = \frac{\partial \mathrm{E}}{\partial \mathrm{t}}$,

Then E and H satisfy $\nabla^2 \mathrm{u} = \frac{\partial^2 u}{\partial t^2}$,

From the above, we are given,

$$\nabla \times (\nabla \times \mathrm{E}) = \nabla \times (-\frac{\partial \mathrm{H}}{\partial \mathrm{t},})$$

$$= -\frac{\partial}{\partial \mathrm{t},}(\nabla \times \mathrm{H})$$

$$= -\frac{\partial}{\partial \mathrm{t},}\left(\frac{\partial \mathrm{E}}{\partial \mathrm{t}}\right)$$

$$= -\frac{\partial^2 E}{\partial t^2}$$

As it is known that $\nabla \times (\nabla \times \mathrm{E}) = -\nabla^2 \mathrm{E} + \nabla(\nabla.\mathrm{E})$

$$= --\nabla^2 \mathrm{E},$$

so that $\nabla^2 \mathrm{E} = \frac{\partial^2 E}{\partial t^2}$,

Similarly $\nabla \times (\nabla \times \mathrm{H}) = \nabla \times (\frac{\partial \mathrm{E}}{\partial \mathrm{t}})$

$$= \frac{\partial}{\partial \mathrm{t},}(\nabla \times \mathrm{E})$$

$$= \frac{\partial}{\partial \mathrm{t},}\left(-\frac{\partial \mathrm{H}}{\partial \mathrm{t},}\right)$$

$$= -\frac{\partial^2 H}{\partial t^2}.$$

But $\nabla \times (\nabla \times \mathrm{H}) = -\nabla^2 \mathrm{H} + \nabla(\nabla.\mathrm{H})$

$= -\nabla^2 H,$

So that $\nabla^2 H = \frac{\partial^2 H}{\partial t^2},$

i.e. E and H satisfy the equation $\nabla^2 u = \frac{\partial^2 u}{\partial t^2}.$

To arrive to Maxwell's equation, we move further as,

$\nabla \times H = \frac{1}{c}\frac{\partial E}{\partial t}$, $\nabla \times E = \frac{1}{c}\frac{\partial H}{\partial t,}$,$\nabla . H = 0$ $\nabla . E = 4\pi\rho,$ where ρ is a function of x, y, z and C is the velocity of light assumed to be constant and which are given by,

E = $-\nabla\emptyset - \frac{1}{c}\frac{\partial A}{\partial t,}$, H=$\nabla \times A$, where A and $\emptyset$ are vectors and scalar potentials respectively, satisfying the equations,

(1) $\nabla . A = \frac{1}{c}\frac{\partial \emptyset}{\partial t,} = 0,$

(2) $\nabla^2 \emptyset - \frac{\partial^2 \emptyset}{\partial t^2} = 4\pi\rho,$

(3) $\nabla^2 A = \frac{1}{c^2}\frac{\partial^2 A}{\partial t^2},$

Maxwell's Equations are given to be,

$\nabla \times H = \frac{1}{c}\frac{\partial E}{\partial t}$ (1)

$\nabla \times E = -\frac{1}{c}\frac{\partial H}{\partial t,}$ (2)

$\nabla . H = 0$ (3)

$\nabla . E = 4\pi\rho$ (4)

The solutions of these equations are given by E = $-\nabla\emptyset - \frac{1}{c}\frac{\partial A}{\partial t,}$ (5)

And H=$\nabla \times A$ (6)

Where A and $\emptyset$ are given by $\nabla . A + \frac{1}{c}\frac{\partial \emptyset}{\partial t,} = 0$ (7)

$\nabla^2 \emptyset - \frac{\partial^2 \emptyset}{\partial t^2} = 4\pi\rho$ (8)

$\nabla^2 A = \frac{1}{c^2}\frac{\partial^2 A}{\partial t^2},$ (Gupta,c, 1980) (9)

Putting H=$\nabla \times A$ from (6), we have,

L.H.S. of (3) above = $\nabla . \nabla \times A$

=$[\nabla \nabla A]$

= 0, by property of scalar triple.

Therefore equation (6) is a solution of (3).
Again putting A=$\nabla \times A$ in (2), we get,

$\nabla \times E = -\frac{1}{c}\frac{\partial H}{\partial t,}(\nabla \times A)$

Or $\nabla \times \left(E + \frac{1}{c}\frac{\partial A}{\partial t,}\right) = 0,$ shows that the bracketed expression is the gradient of some scalar function say $\emptyset$ and therefore$\left(E + \frac{1}{c}\frac{\partial A}{\partial t,}\right) = -\text{grad}\emptyset = -\nabla\emptyset$

i.e.E =$-\nabla\emptyset - \frac{1}{c}\frac{\partial A}{\partial t,}$ (equation) and a solution of (2).

Stoke's Theorem in Space

It state's that if , F is a vector function, which is uniform, finite and continuous, along with its derivatives in any direction, then the tangential line integral of F over any closed surface S bounded by a curve is equal to the normal surface S. Surface integral of curl F over S: i.e.
$\int_c F.\text{dr} = \iint_s (\nabla \times A)$.nds =$\iint_s (\nabla \times A)$.ds.

Where n is the unit normal vector at any point of S in the sense in which right-handed screw would move when rotated in the sense of description C.

Consider a surface S such that its projection on the XY, YZ,and ZX planes are regions bounded by simple closed curves.

Take the equation of surface as f(x,y,z) = 0, i.e. z = $f_1\ (x,y),\ y = f_2\ (x,z),\ x = f_3(y,z)$.

If F =$F_1 i + F_2 j + F_3 z$, we prove that $\iint_s (F_1 i + F_{2_j} + F_3 z)$.nds=$\oint_c F.$ dr,

Then $\iint_s (\nabla \times F_1 i)$.nds=$\oint_c F_1$dx (1)

$\iint_s (\nabla \times F_{2_j})$.nds = $\oint_c F_2$dy (2)

$\iint_s (\nabla \times F_3 z)$.nds = $\oint_c F_3$dz. (3)

Adding (1), (2), and (3) gives $\iint_s (\nabla \times F)$.ds=$\iint_s (\nabla \times F)$.nds=$\oint_c F.$ dr.

If ∅ is continuously differentiable scalar point function, then $\oint_c F.$ dr= $\iint_s (n \times \nabla \emptyset)$.ds.

Stoke's theorem in the plane is sometime called Green Theorem in the plane (Gupta,d, 1980). $\int_c$r $\times\ dr$ = 2$\iint_s$ds, where S is a diaphragm enclosing a circuit C.

Putting F = a $\times\ r$, where a is a constant vector vector , in Stoke's Theorem

i.e. $\int_c F.$dr = $\iint_s (\nabla \times F)$.nds. We get, $\int_c$ (a $\times\ r)$. dr = $\iint_s (\nabla \times (\mathrm{a} \times r))$.nds.

But, $\nabla \times (\mathrm{a} \times r) = \mathrm{a}\nabla.r - (\mathrm{a}.\nabla)r$=3a – a=2a.

∴ $\int_c (\mathrm{a} \times r) \times$. dr = $\iint_s 2a$.ds.

a.$\int_c r$ ×dr = 2a. $\iint_s ds$.

Since, a, is an arbitrary constant vector, $\int_c r$ ×dr = $\iint_s ds$.

$\oint_c F.dr$=0, for every closed curve C is that $(\nabla \times \mathrm{F}) = 0$, identically.

From Stoke's Theorem, $\oint_c F.dr = \iint_s (\nabla \times F).nds = 0$.

Condition is that since $(\nabla \times F) = 0$, then Stoke's theorem at once $\oint_c F.dr = \iint_s (\nabla \times F)$.nds= 0.

Another condition is that, since $\oint_c F.dr = \iint_s (\nabla \times F)$.nds = 0, round any closed curve C, then taking $(\nabla \times F) \neq 0$ at some point P, there will be a region with P an interior point where$(\nabla \times F) \neq 0$ provided,$(\nabla \times F)$ is continuous.

Assuming S to be the surface contained in this region whose unit normal n at each point has the same direction as that of $(\nabla \times F)$ expressing$(\nabla \times F) = \lambda n$, where λ is a positive constant.

Necessary and sufficient condition for $\int_c$F.dr=0 is that $\nabla \times F$=0

By Stoke's Theorem

(a) $\int_c r$.dr =0, as $\int_c r$.dr = $\iint_s (\nabla \times r).nds$ = 0, as $\nabla \times r = 0$.

(b) $\int_c (\emptyset\nabla\emptyset)$.dr =0, as $\nabla \times r).nds = 0$, as $\nabla \times \emptyset\nabla\emptyset = \emptyset\nabla \times (\nabla\emptyset) + (\nabla\emptyset) \times (\nabla\emptyset)$ = 0+0 = 0.

©$\int_c (\emptyset\nabla\varphi)$.dr = -$\int_c (\varphi\nabla\emptyset)$. dr, as$\int_c (\emptyset\nabla\varphi)$. dr =$\iint_s (\nabla\emptyset \times \nabla\varphi).nds$

=-$\iint_s (\nabla\varphi \times \nabla\emptyset)$.nds

and -$\int_c (\varphi\nabla\emptyset)$. dr = $\iint_s \nabla \times (\varphi\nabla\emptyset)).nds$

By Stoke's -$\int_c (\varphi\nabla\emptyset)$. dr, =-$\iint_s (\varphi\nabla \times (\nabla\emptyset) + \nabla\varphi \times \nabla\emptyset)$.nds

=-$\iint_s (\nabla\varphi \times \nabla\emptyset)$.nds, as $\varphi\nabla \times (\nabla\emptyset)$=0

By Stroke's $\oint_c E$.dr =$-\frac{1}{c}\frac{\partial H}{\partial t,}\iint_s H$.ds. Where S is any surface bounded by curve C.

With$\nabla \times E = -\frac{1}{c}\frac{\partial H}{\partial t,}$.

By Stoke's$\oint_c E.\text{dr}=\iint_s(\nabla \times E).\text{n}ds$

$=\iint_s(\nabla \times E).\text{ds}$

$=\iint_s\left(-\frac{1}{c}\frac{\partial \text{H}}{\partial t,}\right).\text{ds} = -\frac{1}{c}\frac{\partial}{\partial t,}\iint_s H.\text{ds},$

The integral being independent of t. Normal surface integral of a vector point function G over every open surface is equal to the tangential line integral of another function F round its boundary, then, G = curl F. Normal integral of a vector point function G is given by $\iint_s$G.ds,

Where S is a surface.

Also, tangential line integral of vector point function F is given by $\int_c F.\text{dr}$,

$\therefore \iint_s G.\text{ds} = \int_c F.\text{dr}$ (1)

Stoke's theorem yields, $\int_c F.\text{dr}=\iint_s(\nabla \times F).\text{nds} = \iint_s(\nabla \times F).\text{ds}$,

$\iint_s G.\text{ds} =\iint_s(\nabla \times F).\text{ds}$ (1),

Which follows that G = ∇ × F,

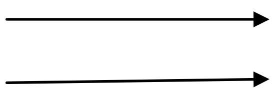
G = curl F

Classification of Vector Fields

Four (4) classes

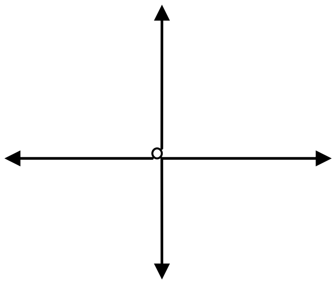
(1) When curl F = 0 and div F = 0;then the field is **Lamellar or Irrotational for curl F = 0,** div F = 0⇒div grad ∅ = 0, i.e. $\nabla^2 \emptyset = 0$. Laplace's Equation, showing the field is **Solenoidal or Incompressible.**

Generally the field is termed, **'Irrotational motion of Incompressible fluid.**



$\nabla.\vec{F} = 0, \; \nabla \times \vec{F} = 0.$

(2) When curl F = 0, but div F ≠ 0, second condition ∇. grad ∅ ≠ 0. The field is **Irrotational motion of Compressible Fluid.**



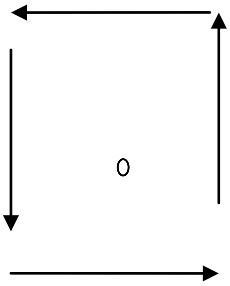
$\nabla \times \vec{F} = 0$ $\nabla . \vec{F} \neq 0$

(3) When curl F ≠ 0, but div F = 0,

F ≠ 0⇒ $\nabla \times (\nabla \times F) \neq 0$ i.e grad div f - $\nabla^2 f$ ≠0, shows that if f is solenoidal, then we must have

Div f = 0, so that grad div f = 0 and as such, $\nabla^2 f$ ≠0. H$\nabla^2 f$ ≠0ence, such a field is

Rotational motion of Incompressible fluid



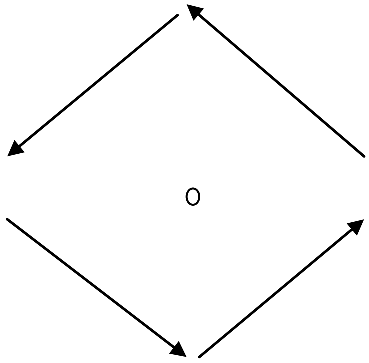
$\nabla \times \vec{F} \neq\neq 0, \nabla . \vec{F} = 0.$

(4) When curl F ≠ 0 and div F ≠ 0.

This type of field is most general and is termed, **Rotational motion of Compressible fluid.** The is made up of two fields namely;

(i) **Lamellar vector field** (i.e. having no curl, but may have only div).

(ii) **Solenoidal vector field,** (i.e. having no div, but may have only curl).



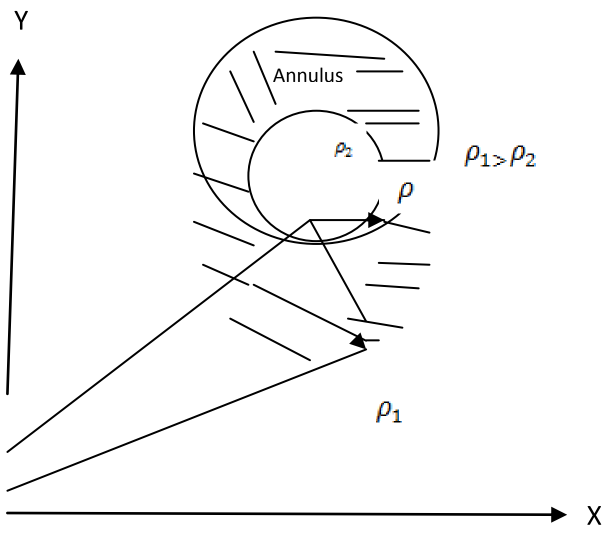
$\nabla \times \vec{F} \neq 0 \quad \nabla . \vec{F} \neq 0$

Where both are existing, that is vector field comprising **Lamellar and Solenoidal field is known as "Helmhotz's Theorem**

Laurent's Force Law,

"States that a force field through concentric circles follows Laurent 's expansion principle which is valid in Taylor series expansion of Fourier Series" (Gupta,e, 1980).

Let there be an annulus between two concentric circles, C_1 and C_2 of centre z = a and radii ρ_1 and ρ_2 ($\rho_1 > \rho_2$), then if f(z), be regular within the annulus between C_1 and C_2 , as well as on the circles and x be any point of the annulus.



F(x) $=\sum_{n=0}^{\infty} a_n(\gamma - a)^n + \sum_{n=1}^{\infty} b_n(\gamma - a)^{-n}$,

Where $a_n = \frac{1}{2\pi i}\int_{C1} \frac{f(z)dz}{(z-a)^{n+1}}$

$$b_n = \frac{1}{2\pi i}\int_{C2} \frac{f(z)dz}{(z-a)^{n-1}}.$$

We have proved in the Taylor's theorem that,

$$\frac{1}{(z-\gamma)} = \frac{1}{(z-a)} + \frac{z-a}{(\gamma-a)^2} + \cdots .. + \frac{(\gamma-a)^n}{(z-a)^n(z-\gamma)}$$

(1)

Interchanging z and γ with each other, we have,

$$\frac{1}{(\gamma-z)} = \frac{1}{(\gamma-a)} + \frac{\gamma-a}{(z-a)^2} + \cdots .. + \frac{(z-a)^n}{(\gamma-a)^n(\gamma-z)} = \frac{1}{(z-\gamma)} \qquad (2)$$

$f(\gamma) = \frac{1}{2\pi i}\int_C \frac{f(z)dz}{(z-\gamma)}$,by Cauchy's integral.

Therefore by making a cross cut, joining any point of the circle C_1 to any point of the circle C_{2}, we get $f(\gamma) = \frac{1}{2\pi i}\int_{C1} \frac{f(z)dz}{(z-\gamma)} - \frac{1}{2\pi i}\int_{c2} \frac{f(z)dz}{(z-\gamma)}$,

$$= \frac{1}{2\pi i}\int_{C1} \frac{f(z)dz}{(z-\gamma)} + \frac{1}{2\pi i}\int_{c2} - \frac{f(z)dz}{(z-\gamma)},$$

$$= \frac{1}{2\pi i}\int_{C1} \frac{1}{(z-\gamma)} f(z)dz + \frac{1}{2\pi i}\int_{c2} - \frac{1}{(z-\gamma)} f(z)dz,$$

Subtituting the values of $\frac{1}{(z-\gamma)}$ and $-\frac{1}{(z-\gamma)}$, from (1) and (2)

$$f(\gamma) = \frac{1}{2\pi i}\int_{C1} [\frac{1}{(z-a)} + \frac{\gamma-a}{(z-a)^2} + \cdots .. + \frac{(\gamma-a)^n}{(z-a)^n(\gamma-z)}] f(z)dz,$$

$$+ \frac{1}{2\pi i}\int_{C2} [\frac{1}{(\gamma-a)} + \frac{z-a}{(\gamma-a)^2} + \cdots .. + \frac{(z-a)^n}{(\gamma-a)^n(z-\gamma)}] f(z)dz$$

Here, term by term integration is possible as two series are uniformly convergent

$$\therefore f(\gamma) = f(a) + \gamma - af^1(a) + \frac{(\gamma-a)^2}{2!} f^{11}(a) + \cdots P_n$$

$$+ \frac{b_1}{\gamma-a} + \frac{b_2}{(\gamma-a)^2} + \frac{b_3}{(\gamma-a)^3} + \cdots + Q_n$$

Where $b_1 = \frac{1}{2\pi i}\int_{C2} f(z)dz$

$b_2 = \frac{1}{2\pi i}\int_{C2} f(z-a)f(z)dz$

b_3----------------------

$b_n = \frac{1}{2\pi i}\int_{C2} f(z-a)^{n-1}f(z)dz$ and

P_n is the remainder after nth term as in Taylor's expansion. It is easy to show (as in Taylor's expansion that $|P_n| \to 0$, as $n \to \infty$

Also $Q_n = \frac{1}{(\gamma-a)^n} \cdot \frac{1}{2\pi i}\int_{C2} \frac{(z-a)^n}{(\gamma-a)}] f(z)dz$

$\because |Q_n| \leq \frac{1}{\gamma^n} \cdot \frac{1}{2\pi}\int_{C2} \frac{|z-a|^n}{|\gamma-a|} | f(z)||dz|$

$\because |Q_n| \leq \frac{{\rho_2}^n}{\gamma^n} \cdot \frac{M^1}{2\pi}\int_{C2} \frac{ds}{\gamma-\rho_2}. \because \ \rho_{1>|}\gamma - a| > \rho_2$ and maximum value of |f(z)|, is M^1

Also, γ-z = γ-a – (z – a) ∴| γ - z| >| |γ-a|- | |z- a|> γ-ρ_2. i.e.$\frac{1}{\gamma-z} > \frac{1}{\gamma-\rho_2}$,

$\leq \frac{{\rho_2}^n}{\gamma^n} \cdot \frac{M^1}{2\pi} \frac{2\pi\rho_2}{\gamma-\rho_2}$,

$\leq ((\frac{\rho_2}{r})^{n+1} \cdot \frac{M^1}{\frac{1-\rho_2}{r}}$,

$\leq$ = 0, as $n \to \infty$ since$\frac{\rho_2}{r}$ <1. Since modulus of any quantity cannot be negative,

∴ Q_n = 0, when $n \to \infty$.

Hence $f(\gamma) = f(a) + \gamma - af^1(a) + \frac{(\gamma-a)^2}{2!}f^{11}(a) + \cdots$

$+\frac{b_1}{\gamma-a} + \frac{b_2}{(\gamma-a)^2} + \frac{b_3}{(\gamma-a)^3} + \cdots +$

= F(x) = $\sum_{n=0}^{\infty} a_n(\gamma-a)^n + \sum_{n=1}^{\infty} b_n(\gamma-a)^{-n}$, (3) Note1; The integrals giving the values of a_n *and* b_n

$$a_n = \frac{1}{2\pi i}\int_{C1} \frac{f(z)\,dz}{(z-a)^{n+1}},$$

$$b_n = \frac{1}{2\pi i}\int_{c2} \frac{f(z)\,dz}{(z-a)^{-n+1}},$$

$$= \frac{1}{2\pi i}\int_{c2} (z-a)^{n-1} f(z)dz$$

Are analytical in everybwhere in the annulus of, C1 and c2.

Note 2; Zeros and Singularity of an analytic complex function.

A zero of an analytic function f(z) is defined to be a value of z = γ, such that f(γ) = 0.

Taylor expansion gives; f(z)=$f(\gamma) + (z-\gamma)f^1(\gamma) + \frac{(z-\gamma)^2}{2!}f^{11}(\gamma) + \cdots + \frac{(z-\gamma)^n}{n!}f^n(\gamma)$.

If $f(\gamma) = 0$ and $f^1(\gamma) \neq 0$, then f(z) is **Simple Zero at**

If $f(\gamma) = f^1(\gamma) = f^{11}(\gamma) = \cdots \ldots f^{n-1}(\gamma) = 0$ and $f^n(\gamma) \neq 0$,

Then the point $z = \gamma$ is Zero of Order n of the $f(z)$.

$.\emptyset(z) = \frac{f^n(\gamma)}{n!} + \frac{z=\gamma}{n+1} + \cdots \ldots \ldots,$

= a non zero and analytic function at $z = \gamma$,

i.e. $f(z) = (z-\gamma)^n[\frac{f^n(\gamma)}{n!} + \frac{(z-\gamma)}{(n+1)!} + \cdots.$

$$= (z-\gamma)^n \emptyset(z) \qquad (4)$$

A point at which function $f(z)$ ceases to be regular (analytic) is termed as **Singular Point of** $f(z)$ and the function $f(z)$ is said to have a **Singularity at this point.**

In case there is no other singularity in the neighbourhood of a singular point $z = \gamma$, the function $f(z)$ is said to have an **Isolated Singularity.**

Laurent's expansion of $f(z) = \sum_{n=0}^{\infty} a_n (z-\gamma)^n + \frac{b_1}{(z-\gamma)} + \frac{b_2}{(z-\gamma)^2} + \cdots .. + \frac{b_m}{(z-\gamma)^m} + \cdots,$

The terms containing b's are termed as **Principal part of the Expansion** at the singularity $z - \gamma$.

This principal exists in three possible ways;

(i) All the coefficients of b's are zero and the function is analytic except at$z - \gamma$. Such a singularity is defined as **non-essential or removable singularity of** $f(z)$.

(ii) If the principal part is an infinite series, then the point f$(z-\gamma)$ is an essential or non-removable singularity of $f(z)$

(iii)If the principal part contains finite number of terms e.g. $\frac{b_1}{(z-\gamma)} + \frac{b_2}{(z-\gamma)^2} + \cdots .. + \frac{b_m}{(z-\gamma)^m};$

$b_{m+1} = b_{m+2} = 0$, then the function$f(z)$has **a pole or Singularity of Order m.** In this case eq. (3) is $f(z) = (z-\gamma)^{-m}[\sum_{n=0}^{\infty} a_n (z-\gamma)^m + b_1(z-\gamma)^{m-1} + b_{2(z-\gamma)^{m-2}} + \cdots .. + b_m]$

$= (z-\gamma)^{-m} \emptyset(z);$ where$\emptyset(z)$ is analytic function at $z = \gamma$. When m=1, the pole is known as **a Simple Pole.**

If an analytic function $f(z)$has a pole of order m at the point $z = \gamma$, then $\frac{1}{f(z)}$, has azero of order m at this point.

Similarly if $\frac{1}{f(z)}$, has a pole of order m, then, $f(z)$ wil have a pole of order m.

It also follows that as zeros are isolated, the pole must be isolated.

Note 3; For complex variables, it is convenient to regard infinity as a point.The point infinity, is considered by putting (z =$\frac{1}{\gamma}$) in $f(z)$ so that the behavior of $f(z)$ at infinity is examined by the behavior of f($\frac{1}{\gamma}$) at $\gamma = 0$.

Consequently $f(z)$ is analytic or has a zero or has a simple pole or has an essential singularity e.t.c. at infinity according as f($\frac{1}{\gamma}$) analytic or has a zero or has a simple pole or has an essential singularity at $\gamma = 0$, e.g if f($\frac{1}{\gamma}$) has a zero of order m at infinity, likewise, if f($\frac{1}{\gamma}$) has a pole of order m at $\gamma = 0$, then $f(z)$ has a pole of order m at infinity.

SUMMARY

The in depth study of this simplified version of electromagnetic fields and waves theories will in no small way help students aspiring into obtaining higher knowledge in this area of Engineering to achieve it with ease. Further work on vectors are developed, after, earlier handling of others including Gauss's divergence theorem, then, Green theorem and identities cum, Poisson's, Laplace's and Maxwell's equations in conjuction with Stoke's and Laurent's force law, are clearly explained in this work.This work will in no small way alleviate the troubles encountered by the learners of Electromagnetic Fields and Waves.

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