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NUMBER OF ZEROS OF A CLASS OF RANDOM ALGEBRAIC POLYNOMIAL

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ABSTRACT

The expected number of crossings of a Random algebraic polynomial f(n)→∞, crosses the line y= mx, when m is any real value and (m^2/n)→0 as n→∞ reduces to only one. We know the expected number of times that a polynomial of degree n with independent normally distributed random real coefficients asymptotically crosses the line y= m x, when m is any real value and (m^2/n)→0 as n→∞. Many authors Kac,[4] Farahmad,[3] Nayak.[7] Rice[9] have investigated the plane symmetric solutions of number of crossings of different polynomials and equations in general relativity. Here, we study the expected number of crossings of a Random algebraic polynomial f (n) →∞, crosses the line y= mx, when m is any real value and (m^2/n) →0 as n→∞ reduces to only one. This theory agrees with the present observational facts pertaining to general relativity. The present paper shows that the expected number of crossings of a Random algebraic polynomial for m >exp(f(n)), where f is any function of n such that f(n)→∞, this expected number of crossings reduces to onlyone.

KEYWORDS AND PHRASES: *Independent identically distributed random variables, random algebraic polynomial, random algebraic equation, real roots.*

1. INTRODUCTION

Consider the algebraic polynomial

$$P(x) = a_0x_0 + a_1x_1 + \cdots + a_nx_n \qquad (1.1)$$

where $a_0, a_1, a_2, \ldots a_n$ is a sequence of independent, normally distributed random variables with mathematical exception μ and variance unity. The set of equations y=P(x) represents a family of curves in the xy-plane, Kac,[3] and Cramer[1] shows that for μ=0 the number of times that this family crosses the line x-axis, on a average is (2/π) log n. Mishra[5] obtained the same average number of crossings when they considered the case of coefficients belongings to the domain of attraction of the normal law with mean μ=0. They also showed that when μ=0 the number ofcrossings reduces by half. Mohanty[7] and Ibragimov[2] studied the number of times that this family of curves crosses the level K=0(√n) (crosses with line y=K) for μ=0 and showed that these numbers decreasesas K increases. He also showed that even in this case for μthe number of crossings reduces by half. Denote by N_m (a,b)= N (a,b) the number of times that this family crosses the line y=mx where m is any constant independent of x and let EN (a,b) be its expectation. For m = 0(√n) an asymptotic value for EN(-∞,∞) was obtained by Mohanty,[7] and Pratihari[8] with which the reader will be assumed to be familiar. As noted in the latter there is a sizeable number of crossings even when the line tends to be perpendicularto the x axis, that is for m (=0 (√n)→∞ as n→∞. In this work we study the case when m is very large compared with n, and show that the number of crossings of this family of curves with such a line reduces to one. We prove.

Theorem- If the coefficients of P(x) in (1.1) are independent normally distributed random variables with mean zero and variance unity, then for any constant m such that $|m| > \exp(f(n))$, where f is any function of n such that f(n) tends to infinity as n tends to infinity, the mathematical expectation of the number of real roots of the equation y=P(x)=mx is asymptotic to one.

2. **Proof of the theorem:-**First we find a lower estimates for EN (-∞,∞). Let m>exp (n,f), then since for $|x| < 1$the polynomial P(x) is convergent, with probability one, for x=1/2, say and n sufficiently large.

$$P(x) = mx < P(1/2) - (1/2)\exp(nf) < 0$$
$$and \text{ also for } x = -1/2$$
$$P(x) = -mx > P(-1/2) + (1/2)\exp(nf) > 0$$

Therefore, by the intermediate value theorem, there exists at least one real root for the function P(x)-mx in the interval (-1/2, ½). Similarly, if m<-exp (nf) we can show that the function P(x)-mx takes on the opposite sign at x=1/2 and x=- 1/2, therefore, there exists at least one real root. Hence EN $(-\infty,\infty) \geq 1$ and we only have to show that the upper limit is one as well. We also note that both aj and –aj (i=0,1,2,….n-1) have standard normal distribution hence changing x to –x leaves the distribution of the coefficients invariant, thus EN (-∞,0)=EN (- ∞,0). So we only have to consider the interval (0,∞). In by using the expected number of level crossings, the Kac-Rice formula[4] for the equation P(x)- mx=0 is found.

$$EN(a,b) = \int_a^b \left[(\Delta^{1/2}/\pi A)\exp\{(-Am^2 + 2m^2Cx - Bm^2x^2)/2D\right] + \left(\sqrt{2/\text{n}}\right)|m(Dx - A)|A^{-a/2}\exp(m_2x^2/2A)erf \{|\text{m}(Dx - A)|(2A\Delta)^{-1/2}\}dx$$

$$= \int_a^b I_1(x)dx + \int_a^b I_2(x)dx \qquad \text{say} \qquad (2.1)$$

Where

$$A = \sum_{i=0}^{n-1}\binom{n-1}{i}x^{2i} = (1+x^2)^{n-1} \qquad (2.2)$$

Since $\frac{d}{dt}(1+t)^{n-1} = \frac{d}{dt}\sum_{i=0}^{n-1}\binom{n-1}{i}t^i = \sum_{i=0}^{n-1}\binom{n-1}{i}it^{i-1}$

$$\Longrightarrow t(n-1)(1+t)^{n-2} = \sum_{i=0}^{n-1} i\binom{n-1}{i}t^i$$

Similarly $\frac{d^2}{dt^2}(1+t)^{n-1} = \sum_{i=0}^{n-1} i(i-1)\binom{n-1}{i}t^{i-2}$

$$\Rightarrow t^2(n-1)(n-2)(1+t)^{n-3} = \sum_{i=0}^{n-1} i(i-1)\binom{n-1}{i}t^i$$

$$\Rightarrow t^2(n-1)(n-2)(1+t)^{n-3} + t(n-1)(1+t)^{n-2} = \sum_{i=0}^{n-1} i^2\binom{n-1}{i}t^i$$

Again $\frac{d^3}{dt^3}(1+t)^{n-1} = \sum_{i=0}^{n-1} i(i-1)(i-2)\binom{n-1}{i}t^{i-3}$

$$\Rightarrow t^3(n-1)(n-2)(n-3)(1+t)^{n-4} = \sum_{i=0}^{n-1}(i^3 - 3i^2 + 2i)\binom{n-1}{i}t^i$$

$$\Longrightarrow t^3(n-1)(n-2)(n-3)(1+t)^{n-4} + 3t^2(n-1)(n-2)(1+t)^{n-3} + t(n-1)(1+t)^{n-2} = \sum_{i=0}^{n-1} i^3\binom{n-1}{i}t^i$$

Again $\frac{d^4}{dt^4}(1+t)^{n-1} = \sum_{i=0}^{n-1} i(i-1)(i-2)(i-3)\binom{n-1}{i}t^{i-4}$

$$\Rightarrow t^4(n-1)(n-2)(n-3)(n-4)(1+t)^{n-5} = \sum_{i=0}^{n-1}(i^4 - 6i^3 + 11i^2 - 6i)\binom{n-1}{i}t^i$$

$$\Longrightarrow t^4(n-1)(n-2)(n-3)(n-4)(1+t)^{n-5} + 6t^3(n-1)(n-2)(n-3)(1+t)^{n-4} + 7t^2(n-1)(n-2)(1+t)^{n-3} + t(n-1)(1+t)^{n-2} = \sum_{i=0}^{n-1} i^4\binom{n-1}{i}t^i$$

Then $B = \sum_{i=0}^{n-1} i^2\binom{n-1}{i}x^{2i-2}$

$$= x^{-2}\sum_{i=0}^{n-1} i^2\binom{n-1}{i}x^{2i}$$

$$= x^{-2}(x^4(n-1)(n-2)(1+x^2)^{n-3} + x^2(n-1)(1+x^2)^{n-2})$$

$$= x^2(n-1)(n-2)(1+x^2)^{n-3} + (n-1)(1+x^2)^{n-2}$$

$$= (n-1)(1+x^2)^{n-3}(x^2(n-2) + 1 + x^2) \qquad (2.3)$$

$$= (n-1)(1+x^2)^{n-3}(\text{n}x^2 - x^2 + 1)$$

C=$\sum_{i=0}^{n-1} i^2(i-1)^2\binom{n-1}{i}x^{2i-4}$

=$x^{-4}\sum_{i=0}^{n-1}(i^4+i^2-2i^3)\binom{n-1}{i}x^{2i}$

=$x^{-4}(x^8(n-1)(n-2)(n-3)(n-4)(1+x^2)^{n-5}+4x^6(n-1)(n-2)(n-3)(1+x^2)^{n-4}+$
$2x^4(n-1)(n-2)(1+x^2)^{n-3})$

=(n-1)(n-2) $(1+x^2)^{n-5}\{x^4(n-3)(n-4)+4x^2(n-3)(1+x^2)+2(1+x^2)^2\}$

=(n-1)(n-2) $(1+x^2)^{n-5}\{n^2x^4-3nx^4+2x^4+4nx^2-8x^2+2\}$

=(n-1)(n-2) $(1+x^2)^{n-5}\{(n-1)(n-2)x^4+4(n-2)x^2+2\}$ (2.4)

D=$\sum_{i=0}^{n-1} i\binom{n-1}{i}x^{2i-1}$

=$x^{-1}\sum_{i=0}^{n-1} i\binom{n-1}{i}x^{2i}$

=$x^{-1}(x^2(n-1)(1+x^2)^{n-2})=x(n-1)(1+x^2)^{n-2}$ (2.5)

E=$\sum_{i=0}^{n-1} i(i-1)\binom{n-1}{i}x^{2i-2}$

=$x^{-2}(\sum_{i=0}^{n-1} i^2\binom{n-1}{i}x^{2i}-\sum_{i=0}^{n-1} i\binom{n-1}{i}x^{2i})$

=$x^{-2}(x^4(n-1)(n-2)(1+x^2)^{n-3})$

= $x^2(n-1)(n-2)(1+x^2)^{n-3}$ (2.6)

F=$\sum_{i=0}^{n-1} i^2(i-1)\binom{n-1}{i}x^{2i-3}$= $x^{-3}(\sum_{i=0}^{n-1} i^3\binom{n-1}{i}x^{2i}-\sum_{i=0}^{n-1} i^2\binom{n-1}{i}x^{2i})$

= $x^{-3}(x^6(n-1)(n-2)(n-3)(1+x^2)^{n-4}+x^4(n-1)(n-2)(1+x^2)^{n-3})$

=$(x^3(n-1)(n-2)(n-3)(1+x^2)^{n-4}+2x(n-1)(n-2)(1+x^2)^{n-3})$

= $(n-1)(n-2)(1+x^2)^{n-4}(\mathrm{n}x^3-x^3+2x)$ (2.7)

and

$$erf(x)=\int_0^x \exp(-y^2)\,dy \qquad (2.8)$$

First we show $\int_0^1 I_1(x)$ dx tends to zero as n→∞. Let a be constant independent of x in the interval (0,1). For $0\le x\le 1-n^{-\alpha}$ and sufficiently large n we have

$$D=\{(n-1)x^{2n+1}-nx^{2n-1}+x\}(1-x^2)^{-2}$$
$$=x(1-x^{2n})(1-x^2)^{-2}+0\{n^{1+\alpha}\exp(-2n^{1-\alpha})\}\ldots\ldots\ldots\ldots\ldots\ldots(2.9)$$

and

$$\mathrm{erf}=\int_0^x \exp(-y^2)dy$$

First we show that $\int_0^1 I_1(x)dx$ tends to zero as n→∞. Let a be a constant independent of x in the interval (0,1). For 0≤x≤1-n^{-a} and all sufficiently large nwe have

$$D=\{(n-1)x^{2n+1}-nx^{2n-1}+x\}(1-x^2)^{-2}$$
$$=x(1-x^{2n})(1-x^2)^{-2}+0\{n^{1+\alpha}\exp(-2n^{1-\alpha})\}\ldots\ldots\ldots\ldots\ldots\ldots(2.9)$$

and

$$B = (1+x^2)(1-x^{2n})(1-x2)^{-3} + 0\{n^{2+a}\exp(-2n^{1-a})\qquad (2.10)$$

From (2.8) and (2.9) we can obtain

$$\Delta = (1-x^{2n})(1-x^2)^{-4} + 0\{n^{2+2n}\exp(-2n^{1-a})\qquad (2.11)$$

Now we choose a=1-{log log $(n)^{10}$} /log n. Then since, for all sufficiently large

n, $n^{2+a}\exp(-2n^{1-a}) = n^{2+2n}\exp(-2\log(n)^{10}\} = n^{-18+a} \rightarrow 0.$

All the error terms that appear in the formulas (2.9) to (2.11) will tend to zero.

Hence from (2.1), (2.9), (2.10), (2.11) and since for all x

$$(1-x^2)^{2/2} - x^2(1-x^2) + x^2(1+x^2)/2 > 1/5$$

we have

$$I_1(x) = \int_0^{1-n^{-a}} (\Delta^{1/2}/wA)\exp\left[-m^2\{(1-x^2)/(1-x^{2n})\}\right]$$
$$\text{x}\ \{(1-x^2)^2/2 - x2(1-x^2)\}$$
$$+x^2(1+x^2)/2\{(1+0\{n^{2+n}\exp(-2n^{1-a})\}dx$$
$$\le (1/\pi)\int_0^{1-n^{-a}} (1-x^2)^{-1}\exp\{-m^2(1-x^2)/5\}dx$$
$$\le (a/2\pi)\exp\{\log\log n\exp - (m^2/2n)\log\ (n)^{10}\}\ \ldots\ldots \qquad (2.12)$$

Now we note that since m>exp (n/logn) the term m^2/n tends to infinity muchfaster than log log n as n→∞, hence from (2.12) we can obtain

$$\int_0^{1-n^{-a}} I_1(x)dx \rightarrow 0\ as\ n \rightarrow \infty. \qquad (2.13)$$

To show that $\int_0^{1-n^{-a}} I_1(x)dx \rightarrow 0\ as\ n \rightarrow \infty.$ we first prove that $(A-2Dx+Bx^2)/\Delta$ is positive for $1-n^{-a} \le x \le 1.$ For all sufficiently large n from (2.2) we have

$$A-2Dx+Bx^2 \ge Bx^2 - 2Dx$$
$$\ge \{n^2x^{2n}(1-x^2)2 - 2nx^{2n+2}(1-x^2)\}$$
$$+x^2(1+x^2)(1-x^{2n})\}(1-x^2) - 3 - n(n+1)$$
$$\ge n^3\{\log(n)^{10}\}^{-3} - 2n^2 > n^2 \qquad (2.14)$$

since $(1-x^2)^3<(2n^{-a})^3$ and $2nx^{2n+2}(1-x^2)\rightarrow$ as n→∞.

Hence from (2.14) and since $\Delta<n^4$ we have

$$(A-2Dx+Bx^2)/\Delta > n^{-2} \qquad (2.15)$$

So from (2.15) and since from $(\Delta^2/a) < (2n-1)^{1/2}(1-x)^{-1/2}$, we have $\int_0^{1-n^{-a}} I_1(x)dx < (2n-1)^{1/2} \exp(-m^2/n^2) \int_0^{1-n^{-a}} (1-x)^{-1/2} dx$

$< 3n^{(1-a)}/2 \exp(-m^2/n^2)$ (2.16)

Which tends to zero as n→∞

In order to find $\int_0^{1-n^{-a}} I_1(x)dx$ we let $y = 1/x$, and divide the interval $0 \le y \le 1$ into three subintervals (0,b), (b,1-(1/nd)) and (1-(1/nd),1) where $d=\{(3/8) \log \log(n)^{1/3}$ and $b=(m^{-2}n \; d \; \log \; n)^{1/(4n-8)}$,. We show that in these three subintervals

$$\int_0^{1-n^{-a}} I_1(x)dx = \int_0^1 y^{-2} I_1(1/y)\,dy = \log\{(1+b)/(1-b)\} \to 0 \text{ as } n \to \infty. \qquad (2.17)$$

For $b \le y \le 1-1/nd$ from (2.2) we have

$$\{(A-2D)/D\} + B/D^2 = 1 + \sum_{i=2}^{n} y^{-2i} + \sum_{i=2}^{n} y^{-2i}(i^2-2i) > n^3/4 \qquad (2.18)$$

and

$$\Delta = \{1-h^2(y)\}(1-y^{2n})2/y^{4n-8}(1-y^2)4 \le n^2/b^{4n-8}(1-y^2)^2$$
$$< 3n^4 d^2/b^{4n-8} \qquad (2.19)$$

To prove that

$\int_0^{\infty} I_1(x)dx \to 0$ *as* n → ∞, *from* (2.2) for y = 1/x we have .

$$(Dx-A)/A^{1/2}\Delta^{1/2} = D^{2n-1}(1-y^2)^{3/2}\{(n-1-ny-y^{2n-1})(1-y^2)^2\}$$
$$- y^{2n-4}\}\{1-h^2(y)\}^{-1/2}(1-y^{2n})^{-1/2} \qquad (2.20)$$

And to find $\int_a^b I_2(x)dx$

$$\text{Let } x^2/A = y^{2n-4}(1-y^2)/(1-y^{2n}) \qquad (2.21)$$

First we let

$0 \le y \le (mn^2)^{-1/(2n-1)}$ then $y \to 0$ as $n \to \infty$, from (2.21)

we have $(Dx - A)/A^{1/2}\Delta^{1/2} < 2my^{2n-1}$ (2.22)

$$\int_0^1 y^{-2} I_2(1/y)\,dy < \int_{m/2n2}^{m/\sqrt{n}} \exp(-u^2/2)du$$

$$\le (m/\sqrt{n} - m^2/2n^2)\exp(-m^2/4n^4) \to 0 \text{ as } n \to \infty. \qquad (2.23)$$

Hence from (2.22) and (2.23) we have

$$\int_1^\infty I_2(x)\,dx \to 0 \text{ as } n \to \infty. \qquad (2.24)$$

Finally from (2.8), (2.13), (2.16), (2.18) and (2.24) we obtain

$EN(0,\infty) \le 1/2$

and since $EN(-\infty,\infty) = 2EN(0,\infty)$ we have proof of the theorem.

RESULT

The asymptotic number of crossings of the polynomial P(x) with line y=mx decreases as m=0(√n) increases. In this paper we proved that when $|m \ge \exp(nf)|$ the number of crossings reduces to one. The behaviour of the number of crossings between these two lines is not known. A subsequent studycould consider the case when (m^2/n) tends to any non zero constant as n tends to infinity and as a guessed target $EN(-\infty,\infty) \sim (1/\pi)$log *n*, which is half the number of crossings when m=0 seems reasonable. (Knowing a rough value for $EN(-\infty,\infty)$is useful in order to sufficient upper and lower bounds for $EN(-\infty,\infty)$ leading to an asymptotic formula). Indeed, the behaviour of $EN(-\infty,\infty)$ for other values of m is also interesting, but it will involve more analysis especially for the $\int_{-\infty}^{\infty} I_2(x)\,dx$ part of $EN(-\infty,\infty)$. The result of the theorem shows that the expected number of crossings of a Random algebraic polynomial form >exp (f(n)), where f is any function of n such that f(n)→∞, this expected number of crossings reduces to only one.

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