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SIMPLIFIED FRACTIONAL FOURIER TRANSFORM AND ITS OPERATORS

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ABSTRACT

Victor Namis provided an elegant generalization of the Fourier transform (FT) to the fractional Fourier transform (FRFT) by deriving the FRFT from the Eigen function of the FT. The idea of using the FRFT for fundamental Signal Processing procedures such filtering, estimation and rotation is particularly interesting applications involving optical information processing. The FRFT has applied to transient motor current signature analysis. Also FRFT has applications in the field of radar system which use for focusing SAR/ISAR images. FRFT can be used in terms of differential equation. Namis solve several Schrodinger equations using this. Now, the researchers define various simplified form of FRFT known as simplified fractional Fourier transform (SFRFT). The reason behind that they are simplest for the digital computation, optical implementation, graded index medium implementation and radar system implementation with the same capability as the conventional FRFT. The aim of this paper is to provide generalization of SFRFT. Also derived some operational formulae as derivative, modulation, scaling property, linearity property and shifting property for simplified fractional Fourier transform.

KEYWORDS: Fourier transforms (FT), fractional Fourier transform (FRFT), simplified fractional Fourier transform (FRFT), derivative, modulation, scaling property, linearity property and shifting property.

INTRODUCTION

The idea of Fourier transform was first suggested by French mathematician Joseph Fourier in 1807. The fractional Fourier transform is a generalization of the ordinary Fourier transforms. Every property and application of the common Fourier transform becomes a special case of the fractional Fourier transform. The fractional Fourier transform was introduced by Wiener[1] as a way to solve certain types of ordinary and partial differential equations arising in quantum mechanics. Unaware of Wiener's work, Victor Namias[2] proposed the fractional Fourier transform also to solve differential equations in quantum mechanics from classical quadratic Hamiltonian. His results were later refined by McBride, and Kerr[3] developed an operational calculus for the transform. The fractional Fourier transform can be used to solve ordinary and partial differential equations as well as fractional and integral equations.

The fractional Fourier transform is generalization of the ordinary Fourier transform.[4] The FRFT implements the so called order parameter α which acts as ordinary Fourier operator. Theα^{th}order fractional Fourier transform represents the α^{th} power of the ordinary Fourier operator. When$\alpha = \pi/2$, we obtain the Fourier transform, while for $\alpha = 0$, we obtain the signal itself. Any intermediate value of $\alpha(0 < \alpha < \pi/2)$ produces signal representation that can be considered as a rotated time-frequency of the signal.[5]

PRELIMINARIES

We define the Fourier transform of a function f(t)

$$F(u) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} e^{-jut} \ f(t)dt$$

The inverse Fourier transform is

$$f(t) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} e^{jut} \ F(u)du$$

The simplified fractional Fourier transform with angle α of a signal f(t) is defined as

$$[\text{SFRFT}(f(t))](u) = (j2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} exp\left(-jut \ + \frac{j}{2} t^2 \, cot\alpha\right) \ f(t)dt.$$

Generalization of Simplified fractional Fourier transform

The Generalization of simplified fractional Fourier transform with parameter α of $f(t)$ denoted by $O^{\alpha}_{F(1)}(\ f(t))$ performs a linear operation given by the integral transform,

$$O^{\alpha}_{F(1)}(f(t)) = F_{\alpha}\{f(t)\}(u)$$

$$[\text{SFRFT f(t) }](\text{u}) = \int_{-\infty}^{\infty} f(t)K_{\alpha}(t,u)\ dt$$

where, $\text{K}_{\alpha}(t,u) = (j2\pi)^{-\frac{1}{2}}e^{\left(-jut+\frac{j}{2}t^2\cot(\alpha)\right)}$

The Test Function Space

An infinitely differential complex valued smooth function on $\emptyset(\text{R}^n)$belongs to $\text{E}(\text{R}^n)$, if for each compact $\text{I} \subset \text{S}_{\alpha}$, where , $\text{S}_{\alpha} = \{t\epsilon\text{R}^n , |t| \leq a , a > 0\}$, $\text{I}\epsilon\text{R}^n$

$\gamma_{\text{E},1}(\emptyset) = \sup_{t\epsilon\text{I}} |\text{D}_t^l\ \emptyset(t)| < \infty$,where $l = 1,2,3 \ldots$

Thus $\text{E}(\text{R}^n)$will denote the space of all $\emptyset\epsilon\ \text{E}(\text{R}^n)$with support contained in S_{α}.

Note that the space E is complete and therefore a Frechlet's space. Moreover, we say that Generalization of simplified fractional Fourier transform if it is a member of E^* , the dual space of E.

Properties

A) Differential Property

Prove That $[SFRFT\ f^{'}(t)](u) = ju[SFRFT\ f(t)](u) - j\cot\alpha\ [SFRFT\ tf(t)](u)$

Proof

$$[SFRFT\ f(t)](u) = \int_{-\infty}^{\infty} f(t)\ k_{\alpha}(t,u)dt$$

$$k_{\alpha}(t,u) = (j2\pi)^{-\frac{1}{2}}e^{\left(-jut+\frac{j}{2}t^2\cot\alpha\right)}$$

$$[SFRFT\ f^{'}(t)](u) = \int_{-\infty}^{\infty} (j2\pi)^{-\frac{1}{2}}e^{\left(-jut+\frac{j}{2}t^2\cot\alpha\right)}f^{'}(t)\ dt$$

$$= (j2\pi)^{-\frac{1}{2}}\left[e^{\left(-jut+\frac{j}{2}t^2\cot\alpha\right)}f(t)\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} (j2\pi)^{-\frac{1}{2}}e^{\left(-jut+\frac{j}{2}t^2\cot\alpha\right)}(-ju + jt\cot\alpha)\,f(t)dt$$

$$= -(j2\pi)^{-\frac{1}{2}}\int_{-\infty}^{\infty} e^{\left(-jut+\frac{j}{2}t^2\cot\alpha\right)}(-ju + jt\cot\alpha)\,f(t)dt$$

$$= (j2\pi)^{-\frac{1}{2}}(ju)\int_{-\infty}^{\infty} e^{\left(-jut+\frac{j}{2}t^2\cot\alpha\right)}\,f(t)dt - j\cot\alpha(j2\pi)^{-\frac{1}{2}}\int_{-\infty}^{\infty} e^{\left(-jut+\frac{j}{2}t^2\cot\alpha\right)}\,tf(t)dt$$

$$= ju[SFRFT\ f(t)](u) - j\cot\alpha\ [SFRFT\ tf(t)](u)$$

B) Linearity Property

Prove that $[SFRFT\ (C_1 f(t) + C_2 g(t))](u) = C_1[SFRFT\ f(t)](u) + C_2[SFRFT\ g(t)](u)$

Proof

If $[SFRFT\ f(t)](u)$ *and* $[SFRFT\ g(t)](u)$ *is generalized simplified fractional fourier transform of* $f(t)$ *and* $g(t)$ *then*

$$[SFRFT\ (C_1 f(t) + C_2 g(t))](u) = (j2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} e^{\left(-jut+\frac{j}{2}t^2 \cot\alpha\right)} (C_1 f(t) + C_2 g(t))dt$$

$$= C_1 (j2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} e^{\left(-jut+\frac{j}{2}t^2 \cot\alpha\right)} f(t)dt + C_2 (j2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} e^{\left(-jut+\frac{j}{2}t^2 \cot\alpha\right)} g(t)\ dt$$

$$= C_1[SFRFT\ f(t)](u) + C_2[SFRFT\ g(t)](u)$$

C) First Shifting Property

Prove that $[SFRFT\ e^{jat} f(t)](u) = [SFRFT\ f(t)](u-a)$

Proof

$$[SFRFT\ e^{jat} f(t)](u) = \int_{-\infty}^{\infty} e^{jat} f(t)\, k_\alpha(t,u)dt$$

$$= (j2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} e^{\left(-jut+\frac{j}{2}t^2 \cot\alpha\right)} e^{jat} f(t)dt$$

$$= (j2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} e^{\left(-jut+\frac{j}{2}t^2 \cot\alpha + jat\right)} f(t)dt$$

$$= (j2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} e^{\left(-j(u-a)t+\frac{j}{2}t^2 \cot\alpha\right)} f(t)dt$$

$$= [SFRFT\ f(t)](u-a)$$

D) Scaling Property

Prove that $[SFRFT\ f(at)](u) = \frac{1}{a}[SFRFT\ f(p)](u)$

Proof

$$[SFRFT\ f(at)](u) = \int_{-\infty}^{\infty} f(at)\, k_\alpha(t,u)dt$$

$$= (j2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} e^{\left(-juat+\frac{j}{2}(at)^2 \cot\alpha\right)} f(at)dt$$

$$putting\ at = p \qquad i.e.\ \ t = \frac{p}{a}$$

$$adt = dp \qquad i.e.\ \ dt = \frac{dp}{a}$$

$$[SFRFT\ f(at)](u) = (j2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} e^{\left(-jup+\frac{j}{2}p^2 \cot\alpha\right)}\ f(p)\frac{dp}{a}$$

$$= \frac{1}{a}(j2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} e^{\left(-jup+\frac{j}{2}p^2 \cot\alpha\right)}\ f(p)dp$$

$$= \frac{1}{a}[SFRFT\ f(p)](u)$$

E) Modulation Property

I) Prove that $[SFRFT\ f(t)\cos at](u) = \frac{1}{2}\{[SFRFT\ f(t)](u-a) + [SFRFT\ f(t)](u+a)\}$

Proof

$$[SFRFT\ f(t)\cos at](u) = (j2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} e^{\left(-jut+\frac{j}{2}t^2 \cot\alpha\right)}\ f(t)\cos at\, dt$$

$$[SFRFT\ f(t)\cos at](u) = (j2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} e^{\left(-jut+\frac{j}{2}t^2 \cot\alpha\right)}\ f(t)\left(\frac{e^{jat}+e^{-jat}}{2}\right)dt$$

$$= \frac{1}{2}(j2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} e^{\left(-jut+\frac{j}{2}t^2 \cot\alpha\right)}\ e^{jat} f(t)dt + \frac{1}{2}(j2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} e^{\left(-jut+\frac{j}{2}t^2 \cot\alpha\right)}\ e^{-jat} f(t)d$$

$$= \frac{1}{2}[SFRFT\ e^{jat} f(t)](u) + \frac{1}{2}[SFRFT\ e^{-jat} f(t)](u)$$

therefore by shifting property

$$= \frac{1}{2}\{[SFRFT\ f(t)](u-a) + [SFRFT\ f(t)](u+a)\}$$

II) Prove that $[SFRFT\ f(t)\sin at](u) = \frac{1}{2j}\{[SFRFT\ f(t)](u-a) - [SFRFT\ f(t)](u+a)\}$

Proof

$$[SFRFT\ f(t)\sin at](u) = (j2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} e^{\left(-jut+\frac{j}{2}t^2 \cot\alpha\right)}\ f(t)\sin at\, dt$$

$$[SFRFT\ f(t)\cos at](u) = (j2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} e^{\left(-jut+\frac{j}{2}t^2 \cot\alpha\right)}\ f(t)\left(\frac{e^{jat}-e^{-jat}}{2j}\right)dt$$

$$= \frac{1}{2j}(j2\pi)^{-\frac{1}{2}}\left\{\int_{-\infty}^{\infty} e^{\left(-jut+\frac{j}{2}t^2 \cot\alpha\right)}\ e^{jat} f(t)dt - \int_{-\infty}^{\infty} e^{\left(-jut+\frac{j}{2}t^2 \cot\alpha\right)}\ e^{-jat} f(t)dt\right\}$$

$$= \frac{1}{2j}\left[SFRFT\ e^{jat} f(t)\right](u) - \frac{1}{2j}\left[SFRFT\ e^{-jat} f(t)\right](u)$$

$therefore\ by\ shifting\ property$

$$= \frac{1}{2j}\{[SFRFT\ f(t)](u-a) - [SFRFT\ f(t)](u+a)\}$$

CONCLUSION

In this paper, generalization of SFRFT provided. Also derived some operational formulae as derivative, modulation, scaling property, linearity property and shifting property for simplified fractional Fourier transform.

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