World Journal of Engineering Research and Technology
WJERT
www.wjert.org
SJIF Impact Factor: 7.029

EXPECTED ZEROS OF RANDOM ORTHOGONAL POLYNOMIALS ON THE REAL LINE

***Pralipta Rout and Dr. P. K. Mishra**

Department of Mathematics Odisha University of Technology and Research Techno Campus, Mahalaxmi Vihar, Naka Gate, Khandagiri, Bhubaneswar, Odisha.

Article Received on 27/06/2024 **Article Revised on 17/07/2024** **Article Accepted on 06/08/2024**

***Corresponding Author**
Pralipta Rout
Department of Mathematics Odisha University of Technology and Research Techno Campus, Mahalaxmi Vihar, Naka Gate, Khandagiri, Bhubaneswar, Odisha.

ABSTRACT

We study the expected number of zeros for random linear combination of orthogonal polynomials with respect to measures supported on the real line. The counting measures of zeros for these random polynomials converge weakly to the corresponding equilibrium measures from potential theory. We quantify this convergence and obtain asymptotic results on the expected number of zeros located in various sets of plane. Random coefficients may be dependent and need not have identical distributions in our results.

KEYWORDS: polynomials, random coefficients, expected number of zeros, uniform distribution, random orthogonal polynomials.

MSC: Primary 30C15; Secondary 30B20, 60B10.

1. Asymptotic equidistribution of zeros

Zeros of polynomials of the form $P_n(z) = \sum_{k=0}^{n} A_k z^k$, where $\{A_k\}_{k=0}^{n}$ are random coefficients, have been studied by Bloch and Pólya, Littlewood and Offord, Erdős and Offord. Kac, Rice, Hammersley, Shparo and Shur, Arnold, and many other authors. It is well known that, under mild conditions on the probability distribution of the coefficients, the majority of zeros of these polynomials accumulate near the unit circumference, being equidistributed in the angular sense. Let $\{Z_k\}_{k=1}^{n}$be the zeros of a polynomial $P_n$of degree n, and define the zero counting measure.

$\tau_n = \frac{1}{n}\sum_{k=1}^{n} \delta_{Z_k}$

Equidistribution for the zeros of random polynomials is expressed via the weak convergence of τ_n to the normalized arclength measure μ_T with probability 1 (abbreviated as a.s. or almost surely). More recent work on the global distribution of zeros of random polynomials include papers of Ibragimov and Zaporozhets[3], Kabluchko and Zaporozhets[4,5], etc. In particular Ibragimov and Zaporozhets[3] proved that if the coefficients are independent and identically distributed, then the condition $E[\log^+|A_o|] < \infty$ is necessary and sufficient for $\tau_n \xrightarrow{w} \mu_T$ almost surely. Here, E[X] denotes the expectation of a random variable X.

A major direction in the study of zeros of random polynomials is related to the ensembles spanned by orthogonal polynomials. The classical case also falls in this category as monomials are orthogonal with respect to $dt/2\pi$ on the unit circumference, which may be used as explanation for clustering of zeros on T. These questions were considered by Bloom and Levenberg[2], Bayraktar[1], the author[7] and others. Many of the mentioned papers used potential theoretic approach to study the limiting zero distribution including that multivariate polynomials.

Let A_k, k = 0, 1, 2,……., be complex valued random variables. We state results on the asymptotic zero distribution under general assumptions that do not require independence or identical distribution of random coefficients. Let the distribution function of $|A_k|$ be defined by $F_k(x) = P(\{|A_k| \le x\})$, $x \in \mathbb{R}$. Suppose that there is $N \in \mathbb{N}$, a decreasing function $f:[a, \infty) \to [0, 1]$, a>1, and increasing function $g:[0, b] \to [0, 1]$, 0<b<1, such that

$$\int_a^\infty \frac{f(x)}{x}dx < \infty \text{ and } 1 - F_k \le f(x), \forall x \in [a, \infty), \quad (1.1)$$

$$\int_a^\infty \frac{g(x)}{x}dx < \infty \text{ and } F_k \le g(x), \forall x \in [0, b], \quad (1.2)$$

Hold for all $k \ge N$.

If F(x)is the distribution function of |X|, where X is a complex random variable, then

$$E[\log^+|X|] < \infty \Leftrightarrow \int_a^\infty \frac{1-F(x)}{x}dx < \infty, \ a \ge 0,$$

And

$$E[\log^-|X|] < \infty \Leftrightarrow \int_0^b \frac{F(x)}{x}dx < \infty, \ b \ge 0,$$

Hence, when all random variables $|A_k|$, k = 0,1,2,……, are identically distributed, assumptions (1.1)-(1.2) are equivalent to $E[|\log|A_0||] < \infty$.

Define a sequence of orthonormal polynomials $\{P_n\}_{n=0}^{\infty}$ with respect to a positive Borel measure ν supported on E ⊂ ℝ and possessing finite moments, where $p_n(z) = c_n z^n + \ldots\ldots.$and $c_n > 0$. We consider orthonormal polynomials with respect to a general measure with compact support, and the Freud polynomials orthonormal over E = ℝ with respect to an exponential weight. The goal of this paper is the study of zeros for the ensembles of random orthogonal polynomials.

$$P_n(z) = \sum_{k=0}^{n} A_k\, p_k(z) \qquad (1.3)$$

Our first ensemble is spanned by orthogonal polynomials with respect to a measure $\nu \in \mathbf{Reg}$ with compact support E ⊂ ℝ that is regular in the sense of logarithmic potential theory. We use the notation **Reg** for the class of measures regular in the sense of Stahl, Totik and Ullman. In particular, the class **Reg** is characterized by the following asymptotic property for the leading coefficients of orthonormal polynomials p_n:

$$\lim_{n\to\infty} c_n^{1/n} = 1/\mathrm{cap}(E)$$

Where $\mathrm{cap}(E)$ is the logarithmic capacity of E. Since E is a regular set in our case, the above limit is equivalent to the following:

$$\lim_{n\to\infty} \|p_n\|_E^{1/n} = 1 \qquad (1.4)$$

Where $\|p_n\|_E$ denotes the supremum norm of p_n on E. We first show that the counting measures of zeros converge weakly to the equilibrium measure of E denoted by μ_E, which is a positive unit Borel measure supported on E.

Theorm 1.1. Suppose that the measure $\nu \in \mathbf{Reg}$ defining the orthonormal polynomials $\{p_k\}_{k=0}^{\infty}$ has compact support E ⊂ ℝ that is regular in the sense of logarithmic potential theory. If the random coefficients $\{A_k\}_{k=0}^{\infty}$ satisfy (1.1)-(1.2), then the zero counting measures of the random orthogonal polynomials (1.3) converge almost surely to μ_E as n→∞.

The condition of regularity for ν is standard in the theory of general orthogonal polynomials. If E □ ℝ is a finite union of compact intervals, and if dv(x) = w(x)dx with w(x) > 0 a.e. on E, then $\nu \in$ **Reg.**

If E = [a, b] □ ℝ then it is well known that

$$d\mu_{[a,b]}(x) = \frac{dx}{\pi\sqrt{(b-x)(x-a)}},\ x\in(a,b),$$

which is the Chebyshev (arcsin) distribution. More generally, if $E = \cup_{l=1}^{N}[a_l, b_l]$ for N ≥ 2, where $a_1 < b_1 < a_2 < b_2 < \ldots\ldots < a_N < b_N$ are real numbers, then there exist $y_l \in (b_l, a_{l+1})$, l=1,…..,N− 1, such that the equilibrium measure of E is given by

$$d\mu_E(x) = \frac{\prod_{l=1}^{N-1}|x-y_l|\,dx}{\pi\sqrt{\prod_{l=1}^{N}|x-a_l||x-b_l|}},\ x\in \cup_{l=1}^{N}(a_l, b_l)$$

Theorem 1.1 allows to find asymptotics for the expected number of zeros in various sets.

Corollary 1.2. Suppose that all assumptions of Theorm 1.1 hold, and denote the number of zeros for the random orthogonal polynomials (1.3) in a set S □ ℂ by N_n(S). If S □ ℂ is a compact set satisfying $\mu_E(\partial S) = 0$, then

$$\lim_{n\to\infty}\frac{1}{n}E[N_n(S)] = \mu_E(S)$$

A typical example of a set S is given by any rectangle of the form {x + iy ∈ ℂ: a ≤ x ≤ b, c ≤ y ≤ d}, where a < b and c < 0 < d are real numbers. Note that Corollary 1.2 is not directly applicable to sets S □ E. Lubinsky, the author and Xie[6] proved for random orthogonal polynomials with real Gaussian coefficients that

$$\lim_{n\to\infty}\frac{1}{n}E[N_n(S)] = \frac{1}{\sqrt{3}}\mu_E(S),\ S \text{ □ } E.$$

We now turn to the second family of orthogonal polynomials related to the Freud (exponential) weights

$$W(x) = e^{-ic|x|^{\lambda}},\ x \in \mathbb{R}, \tag{1.5}$$

Where c > 0 and λ > 1 are constants. It is customary to define the orthogonality relation in this case by

$$\int_{\mathbb{R}} p_n(x)p_m(x)W^2(x)dx = \delta_{mn}.$$

We need to introduce a scaling parameter to study the asymptotic distribution of zeros for random Freud orthogonal polynomials. Define the constants.

$\Upsilon_\lambda = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{\lambda}{2})}{2\Gamma(\frac{\lambda+1}{2})}$ and $a_n = \gamma_\lambda^{1/\lambda}\ c^{-1/\lambda} n^{1/\lambda}$,

and consider the contracted version of P_n from (1.3):

$P_n^* := P_n(a_n s)$, n ∈ N. (1.6)

Consider the normalized zero counting measure $\tau_n^* = \frac{1}{n}\sum_{k=1}^{n} \delta_{Z_k}$ for the scaled polynomial $P_n^*(S)$ of (1.6) where $\{Z_k\}_{k=1}^{n}$ are its zeros, and δ_z denotes the unit point mass at z. The limiting measure for τ_n^* is described by the Ullman distribution

$d\mu_\lambda(s) = \left(\frac{\lambda}{\pi}\int_{|s|}^{1} \frac{y^{\lambda-1}}{\sqrt{y^2 - s^2}}\, dy\right) ds$, s ∈ $[-1,1]$.

Note that μ_λ is the weighted equilibrium measure for the weight $w_\lambda(x) = e^{-\gamma_\lambda |x|^\lambda}$ on ℝ.

Theorm 1.3. If the random coefficients $\{A_k\}_{k=0}^{\infty}$ satisfy (1.1)-(1.2), and $\{p_k\}_{k=0}^{\infty}$ are Freud orthogonal polynomials, then the normalized zero counting measures τ_n^* for the scaled polynomials $P_n{}^*(s)$ of (1.6) converge weakly to μ_λ with probability one.

Zeros distribution for random orthogonal polynomials with varying weights was studied by Bloom and Levenberg (see Section 6 of[2]). It might be possible to extend the above result to the class of superlogarithmic weights considered in.[2] As before, we find asymptotics for the expected number of zeros in sets that do not have significant overlap with the real line.

Corollary 1.4. Let the number of zeros for the polynomials (1.6) in a set S ⊂ ℂ be denoted by $N_n^*(S)$. If the assumptions of Theorem 1.3 are satisfied, and S ⊂ ℂ is a compact set such that $\mu_\lambda(\partial S) = 0$, then

$\lim_{n\to\infty} \frac{1}{n} E[N_n^*(S)] = \frac{1}{\sqrt{3}}\mu_\lambda(S)$, S ⊂ $(-1, 1)$.

2. Expected Number of Zeros: Union of Compact Intervals

The main results of this paper provide quantitative estimates for the weak convergence of the zero counting measures of random orthogonal polynomials (1.3) to the corresponding equilibrium measure. In In particular, we study the expected deviation of the normalized counting measure of zeros τ_n from the equilibrium measure μ_E on certain sets, which is often called the discrepancy between those measures. We use a discrepancy result of Blatt and Grothmann for deterministic polynomials to obtain the following.

Theorm 2.1. Suppose that E □ ℝ is a finite union of compact intervals, and that the polynomials $\{p_k\}_{k=0}^{n}$ are orthonormal over E with respect to a weight w ≥ 0. Let $\{A_k\}_{k=0}^{n}$ be complex random variables satisfying $E[|A_k|^t] < \infty$, k = 0,…….,n, for a fixed t ∈ (0, 1], and $E[|\log|A_n||] > -\infty$. If I □ ℝ is any interval and S(I) := $\{z \in \mathbb{C} : \text{Re } z \in I\}$, then we have for the zero counting measure τ_n of the random polynomials (1.3) that

$$E[|(\tau_n - \mu_E)(S(I))|] \leq 8 \left[\frac{1}{n}\left(\frac{1}{t}\log(\textstyle\sum_{k=0}^{n} E[|A_k|^t]) + \log\max_{0\leq k\leq n}\|p_k\|_E + B - E[|\log|A_n||]\right)\right]^{1/2}, \quad (2.1)$$

Where B is a constant that depends only on E and w.

One can replace the strip S(I) with other sets containing the interval I, e.g., with rectangles or sets bounded by the level curves of the Green function for ℂ\E. This gives essentially the same estimate as in (2.1), but with constant 8 replaced by a different one.

In order to obtain more effective bounds, we consider the orthonormal polynomials p_n that satisfy

$$\|p_n\|_E = O(n^p) \text{ as } n \to \infty, \quad (2.2)$$

for a fixed positive constant p. This condition holds for many important classes of weights, as discussed below.

Corollary 2.2. Under the assumptions of Theorem 2.1, suppose that fort ∈ (0, 1] we have

$$\sup_{n\geq 0} E[\|A_n\|^t] < \infty \quad (2.3)$$

And

$$\lim_{n\to\infty} \inf E[\log|A_n|] > -\infty \quad (2.4)$$

If the orthogonal polynomials p_n satisfy (1.4), then

$$\lim_{n\to\infty} E[|(\tau_n - \mu_E)(S(I))|] = 0 \quad (2.5)$$

$$\lim_{n\to\infty} E[|(\tau_n - \mu_E)(S(I))|] = O\left(\frac{\sqrt{\log n}}{n}\right) \text{ as } n \to \infty. \quad (2.6)$$

Since τ_n(S) = N_n(S) is the number of zeros for P_n in S, (2.6) can be restated as

E[$N_n(S(I))$] = nμ_E(S(I)) + O$\left(\sqrt{n\log n}\right)$.

It is well known from the original work of Erdős and Turán that the order $\sqrt{\log n/n}$ of the right hand side in (2.6) is optimal in the deterministic case. Growth condition (2.2) holds true for polynomials orthogonal with respect to the generalized Jacobi weights of the form w(x) = v(x) $\prod_{j=1}^{J}|x-x_j|^{\alpha_j}$, where v(x) ≥ c > 0 a.e. on E. Other classes of weights that generate orthogonal polynomials satisfying (2.2) can be obtained from a Nikolskii type inequality for algebraic polynomials Q_n:

$\|Q_n\|_E \leq Cn^p\left(\int_E |Q_n(x)|^2 w(x)dx\right)^{1/2}$,

where the integral on the right is equal to 1 for p_n.

We conclude this section by showing that the expected number of zeros for random orthogonal polynomials located at a positive distance from E is essentially negligible.

Theorm 2.3. Suppose that the assumptions of Theorem 2.1 are satisfied. If S □ $\overline{\mathbb{C}}/E$ is any closed set, then

$$E[\tau_n(S)] \leq \frac{1}{bn}\left(\left(\frac{1}{t}\log\sum_{k=0}^{n}E[|A_k|^t]\right) + \log\max_{0\leq k\leq n}\|p_k\|_E + \text{B} - \text{E}\,[\log|A_n|]\right) \quad (2.7)$$

Where B depends only on E and w, and b > 0 depend only on E and S. In fact, b: = $min_S\, g_E$ where g_E is the Green function for the complement of E with pole at infinity.

Uniform assumptions give the following quantitative results for large n ∈ ℕ.

Corollary 2.4. If under the assumptions of Theorem 2.3 equations (2.2), (2.3) and (2.4) hold true, then

$$E[\tau_n(S)] = O\left(\frac{\log n}{n}\right) \text{ as } n \to \infty \quad (2.8)$$

$$E[N_n(S)] = O(\log n) \text{ as } n \to \infty \quad (2.9)$$

It is now easy to modify the vertical strip S(I) of Theorm 2.1 into rather arbitrary set containing the interval I, by removing parts of this strip that are separated from I by a positive distance. Indeed, the expected number of zeros in those parts is of the order $\log n$, which is absorbed by the right hand side terms in (2.1) and (2.6).

3. Expected number of zeros: Freud weights on ℝ

We study the same questions on the expected number of zeros for random orthogonal polynomials as in the previous section, but the spanning polynomials p_n are now orthogonal with respect to a Freud weight (1.5) defined on the whole real line. Theorem 1.3 shows that the normalized zero counting measures $\tau_n{}^*$ for the scaled polynomials $P_n{}^*$(S) of (1.6) converge weakly to the Ullman distribution μ_λ with probability one. Our next result gives an estimate for the rate of this convergence in terms of expected deviation of $\tau_n{}^*$ from μ_λ on certain test sets. Note that our assumptions on random coefficients are the same as in the previous section, and are different from those of Theorem 1.3.

Theorm 3.1. Let $\{A_k\}_{k=0}^n$ be complex random variables satisfying $E[|A_k|^t] < \infty$, k=0,……,n, for a fixed t ∈ (0,1], and $E[\log|A_n|] > -\infty$. If I □ ℝ is any interval and S(I) :=$\{z \in c\,; Re\ z\ \in I\}$, then the zero counting measures $\tau_n{}^*$ of the random polynomials (1.6) satisfy

$$E[|(\tau_n{}^* - \mu_\lambda)(S(I))|] \le C_1\left[\frac{1}{n}\left(\frac{1}{t}\log\left(\textstyle\sum_{k=0}^{n} E[|A_k|^t]\right)\right) + C_2\log n - E[\log|A_n|]\right]^{1/2}, \quad (3.1)$$

Where C_1, $C_1 > 0$ depend only on the constants c > 0 and $\lambda > 1$ in the weight W of (1.5).

Corollary 3.2. If under the assumptions of Theorm 3.1 we have

$\sup_{n\ge 0} E[|A_n|^t] < \infty$ (3.2)

$\lim_{n\to\infty} \inf E[\log|A_n|] > -\infty,$ (3.3)

$$E[|(\tau_n{}^* - \mu_\lambda)(S(I))|] = O\left(\sqrt{\frac{\log n}{n}}\right) \text{ as } n \to \infty \quad (3.4)$$

Letting $N_n{}^*$(S) be the number of zeros for $P_n{}^*$ in a set S □ ℂ, we give an alternative form for (3.4):

$$E[N_n{}^*(S(I))] = n\mu_\lambda(S(I)) + O\left(\sqrt{n\log n}\right).$$

It is also possible to establish analogs of Theorem 2.3 and Corollary 2.4 for closed sets S □ $\bar{\mathbb{C}}$ \ $[-1,1]$,but we do not develop this direction.

Proofs

4.1. Proofs for Section 1

We need to first develop results on the n-th root limits of random coefficients. These results are mostly known, but we include the proofs for convenience. Let $\{A_n\}_{n=0}^{\infty}$ be a sequence of complex valued random variables, and Let F_n be the distribution function of $|A_n|$, $n \in \mathbb{N}$. Our assumptions for A_n below are stated in (1.1) and (1.2).

Lemma 4.1. If there is $N \in \mathbb{N}$ and a decreasing function $f : [a,\infty) \to [0,1]$, $a > 1$, such that

$\int_a^{\infty} \frac{f(x)}{x} dx < \infty$ and $1 - F_n(x) \le f(x)$, $\forall\, x \in [a,\infty)$,

holds for all $n \ge N$, then

$$\lim_{n\to\infty} \sup |A_n|^{1/n} \le 1 \quad \text{a.s.} \qquad (4.1)$$

Furthermore, if there is $N \in \mathbb{N}$ and an increasing function $g: [0,b] \to [0,1]$, $0 < b < 1$, such that

$\int_a^{\infty} \frac{g(x)}{x} dx < \infty$ and $F_n(x) \le g(x)$, $\forall\, x \in [0,b]$,

holds for all $n \ge N$, then

$$\lim_{n\to\infty} \inf |A_n|^{1/n} \ge 1 \text{ a.s.} \qquad (4.2)$$

Hence, if both assumptions (1.1) and (1.2) are satisfied for $\{A_n\}_{n=0}^{\infty}$, then

$$\lim_{n\to\infty} |A_n|^{1/n} = 1 \qquad \text{a.s.} \qquad (4.3)$$

The almost sure limits of (4.1)-(4.3) follow from the first Borel-Cantelli lemma.

Lemma (Borel-Cantelli Lemma). Let $\{\varepsilon_n\}_{n=1}^{\infty}$ be a sequence of arbitrary events. If $\sum_{n=1}^{\infty} \mathbb{P}(\varepsilon_n) < \infty$ then (ε_n occurs infinitely often) = 0.

Proof of Lemma 4.1. We first prove (4.1). For any fixed $\epsilon > 0$, define events $\varepsilon_n = \{|A_n| > e^{\epsilon n}\}$, $n \in \mathbb{N}$. Using the first assumption and letting $m := \max\left(N, \left\lfloor \frac{1}{\epsilon} \log a \right\rfloor\right) + 2$, we obtain $\sum_{n=m}^{\infty} \mathbb{P}(\varepsilon_n) = \sum_{n=m}^{\infty}\left(1 - \mathbb{P}(\{|A_n| \le e^{\epsilon n}\})\right) = \sum_{n=m}^{\infty}\left(1 - F_n(e^{\epsilon n})\right) \le \sum_{n=m}^{\infty} f(e^{\epsilon n}) \le \int_{m-1}^{\infty} f(e^{\epsilon t}) dt \le \frac{1}{\epsilon} \int_a^{\infty} \frac{f(x)}{x} dx < \infty$.

Hence $\mathbb{P}(\varepsilon_n$ occurs infinitely often) = 0 by the first Borel-Cantelli lemma, so that the complementary event ${\varepsilon_n}^c$ must happen for all large n with probability 1. This means that $|A_n|^{1/n} \leq e^{\epsilon}$ for all sufficiently large n∈N almost surely. We obtain that

$\lim_{n\to\infty} \sup|A_n|^{1/n} \leq e^{\epsilon}$ a.s.,

and (4.1) follows because $\epsilon > 0$ may be arbitrarily small.

The proof of (4.2) proceeds in a similar way. For any given $\epsilon > 0$, we set $\varepsilon_n = \{|A_n| \leq e^{-\epsilon n}\}, n \in \mathbb{N}$. Using the second assumption and letting m: $=\max\left(N, \left\lfloor \frac{-1}{\epsilon} \log b \right\rfloor\right) + 2$, we have

$$\sum_{n=m}^{\infty} \mathbb{P}(\varepsilon_n) = \sum_{n=m}^{\infty} F_n(e^{-\epsilon n}) \leq \sum_{n=m}^{\infty} g(e^{-\epsilon n}) \leq \int_{m-1}^{\infty} g(e^{\epsilon t})dt \leq \frac{1}{\epsilon}\int_0^b \frac{g(x)}{x}dx < \infty.$$

Hence $\mathbb{P}(\varepsilon_n\ i.o.) = 0$, and $|A_n|^{1/n} > e^{-\epsilon}$ holds for all sufficiently large n ∈ ℕ almost surely. We obtain that

$\lim_{n\to\infty} \inf|A_n|^{1/n} \geq e^{-\epsilon}$ a.s.,

And (4.2) follows by letting $\epsilon \to 0$.

We also need the following simple consequence of (4.3).

Lemma 4.2. If (1.1) and (1.2) hold for the coefficients $\{A_n\}_{n=0}^{\infty}$ of random polynomials, then

$\lim_{n\to\infty} \max_{0\leq k\leq n}|A_k|^{1/n} = 1$ a.s. (4.4)

Proof. We deduce (4.4) from (4.3). Let w be any elementary event such that

$\lim_{n\to\infty}|A_n(w)|^{1/n} = 1,$

which holds with probability one. We immediately obtain that

$\lim_{n\to\infty}\inf \max_{0\leq k\leq n}|A_k(w)|^{1/n} \geq \lim_{n\to\infty} \inf|A_n(w)|^{1/n} = 1.$

On the other hand, elementary properties of limits imply that

$\lim_{n\to\infty}\sup \max_{0\leq k\leq n}|A_k(w)|^{1/n} \leq 1.$

Indeed, for any $\epsilon > 0$ there exists $n_\epsilon \in \mathbb{N}$ such that $\lim_{n\to\infty}|A_n(w)|^{1/n} \leq 1+\epsilon$ for all $n \geq n_\epsilon$ by (4.3). Hence,

$\max_{0\leq k\leq n}|A_k(w)|^{1/n} \leq \max\left(\max_{0\leq k\leq n_\epsilon}|A_k(w)|^{1/n}, 1+\epsilon\right)$ as $n \to \infty$, And the result follows by letting $\epsilon \to 0$.

We state a special case of Theorem 2.1 from Blatt, Saff and Simkani, which is used to prove Theorem 1.1.

Theorem BSS. Let E □ ℂ be a compact set of positive capacity cap(E), with empty interior and connected complement. If a sequence of monic polynomials $M_n(z)$ of degree n satisfy

$\lim_{n\to\infty} sup\|M_n\|_E^{1/n} \leq$ cap(E), (4.5)

then the zero counting measures τ_n of $M_n(z)$ converge weakly to μ_E as $n \to \infty$.

Proof of Theorem 1.1. Let the leading coefficient of the orthogonal polynomial $p_n(z)$ be c_n > 0. Then

$p_n(z) = \sum_{k=0}^{n} A_k\, p_k(z) = A_n c_n z^n + \ldots\ldots,\ \ n \in \mathbb{N}$.

Theorem 1.1 is proved by applying Theorem BSS to the monic polynomials

$M_n(z) := \frac{P_n(z)}{A_n c_n}$, $n \in \mathbb{N}$.

We first estimate the norm

$\|p_n\|_E \leq \sum_{k=0}^{n} |A_k|\ \|p_k\|_E \leq (n+1)\ \max_{0\leq k\leq n}|A_k|\ \max_{0\leq k\leq n}\|p_k\|_E$.

Note that (1.4) implies by an elementary argument (already used in the proof of Lemma 4.2) that

$\lim_{n\to\infty} \sup(\max_{0\leq k\leq n}\|p_k\|_E)^{1/n} \leq 1$.

Combining this fact with (4.4), we obtain that

$\lim_{n\to\infty} sup\|P_n\|_E^{1/n} \leq 1$ a.s. (4.6)

Since $\nu \in$ **Reg,** the leading coefficients c_n of the orthonormal polynomials p_n satisfy

$\lim_{n\to\infty} c_n^{\frac{1}{n}} = 1/\text{cap}(E)$. (4.7)

Applying (4.2) and (4.7), we obtain that

$\lim_{n\to\infty} \inf|A_n c_n|^{1/n} \geq 1/\text{cap}(E)$ a.s.

Using this together with (4.6), we conclude that (4.5) holds for our monic polynomials $M_n = p_n/(A_n c_n)$ with probability one.

Proof of Corollary 1.2. Theorem 1.1 implies that the counting measures τ_n converge weakly to μ_E with probability one. Since $\mu_E(\partial S) = 0$, we obtain that τ_n | S converges weakly to τ_n | S with probability one. In particular we have that the random variables τ_n(S) $\rightarrow$ μ_E(S) a.s. Hence this convergence holds in L^p sense by the Dominated Convergence Theorem, as τ_n(E) are uniformly bounded by 1. It follows that

$$\lim_{n\to\infty} E[|(\tau_n(S) - \mu_E(S))|] = 0$$

For any compact set E such that $\mu_E(\partial S) = 0$, and

$$|E[|(\tau_n(S) - \mu_E(S))|]| \leq \mathrm{E}[|(\tau_n(S) - \mu_E(S))|] \to 0 \quad \text{as } n \to \infty.$$

But E[$\tau_n(S)$] = E$E[N_n(S)]/n$, which immediately gives the result.

we call a sequence of monic polynomials $\{Q_n\}_{n=1}^{\infty}$, with deg(Q_n) = n, asymptotically extremal with respect to the weight w_λ if it satisfies

$$\lim_{n\to\infty} \|w_\lambda{}^n Q_n\|_R{}^{1/n} = e^{-F_\lambda},$$

Where $\|.\|_R$ is the supremum norm on ℝ and $F_\lambda = \log 2 + 1/\lambda$ is the modified Robin constant corresponding to w_λ. It states that any sequence of such asymptotically extremal monic polynomials have their zeros distributed according to the measure μ_λ. More precisely, the normalized zero counting measures of Q_n converge weakly to μ_λ. Denote the leading coefficient of the Freud orthonormal polynomial p_n by c_n. Recall the scaling paremeter $a_n = \gamma_\lambda^{\frac{1}{\lambda}} c^{-\frac{1}{\lambda}} n^{\frac{1}{\lambda}}$ used to define the polynomials $P_n{}^*$ in (1.6). We show that the monic polynomials

$Q_n{}^*(x) := P_n{}^*(x)/(A_n c_n a_n{}^n)$, n ∈ ℕ,

are asymptotically extremal in the above sense with probability one, so that the result of Theorem 1.3 follows.

Using orthogonality, we obtain for the polynomials defined in (1.3) that

$$\int_{-\infty}^{\infty} |P_n(x)|^2 W^2(x)\, dx = \sum_{k=0}^{n} |A_k|^2.$$

Hence,

$\max_{0\le k\le n}|A_k| \le \left(\int_{-\infty}^{\infty}|P_n(x)|^2\, W^2(x)\, dx\right)^{1/2} \le (n+1)\max_{0\le k\le n}|A_k|$,

and Lemma 4.2 implies that

$\lim_{n\to\infty}\left(\int_{-\infty}^{\infty}|P_n(x)|^2\, W^2(x)\, dx\right)^{1/(2n)} = \lim_{n\to\infty}(\max_{0\le k\le n}|A_k|)^{1/n} = 1$

with probability one. Applying the Nikolskii-type inequalities, we obtain that the same holds for the supremum norm:

$\lim_{n\to\infty}\|w_\lambda{}^n Q_n{}^*\|_{\mathbb{R}}{}^{1/n} = \lim_{n\to\infty}\|P_n W\|_{\mathbb{R}}{}^{1/n} = 1$

with probability one. Recall that the leading coefficients of the Freud orthonormal polynomials p_n satisfy

$\lim_{n\to\infty} c_n{}^{1/n}\, n^{1/\lambda} = 2c^{1/\lambda}\gamma_\lambda{}^{-1/\lambda} e^{1/\lambda}$

We also use below that $\lim_{n\to\infty}|A_n|^{1/n} = 1$ with probability one by (4.3). It follows that $\lim_{n\to\infty}\|w_\lambda{}^n Q_n^*\|_{\mathbb{R}}{}^{1/n} = \lim_{n\to\infty}\|w_\lambda{}^n Q_n^*\|_{\mathbb{R}}{}^{1/n} \quad \lim_{n\to\infty}|A_n c_n|^{-1/n} \quad |a_n|^{-1} = \lim_{n\to\infty}\left(c_n{}^{1/n} n^{1/\lambda} c^{-1/\lambda}\gamma_\lambda{}^{1/\lambda}\right)^{-1} = \left(2c^{1/\lambda}\gamma_\lambda{}^{-1/\lambda} e^{-1/\lambda}\gamma_\lambda{}^{1/\lambda}\right)^{-1} = e^{-(\log 2 + 1/\lambda)} = e^{-F_\lambda}$.

Proof of Corollary 1.4. This result is proved in exactly the same way as Corollary 1.2, only replacing τ_n with τ_n^*, and μ_E with μ_λ. The weak convergence of τ_n^* to μ_λ with probability one is provided by Theorm 1.3.

4.2. Proofs for Section 2

We need the following consequence of Jensen's inequality.

Lemma 4.3. If A_k, k = 0,….., n, are complex random variables satisfying $E[|A_k|^t] < \infty$, k = 0,….., n, for a fixed t ∈ (0, 1], then

$E[\log \sum_{k=0}^{n}|A_k|] \le \frac{1}{t}\log(\sum_{k=0}^{\infty} E[|A_k|^t])$. (4.8)

Proof. We first state an elementary inequality. If $x_i \ge 0$, I =0,……., n, and $\sum_{i=0}^{n} x_i = 1$, then

$\sum_{i=0}^{n}(x_i)^t \ge \sum_{i=0}^{n} x_i = 1$

For t ∈ (0, 1]. Applying this inequality with $x_i = |A_i|/\sum_{k=0}^{n}|A_k|$, we obtain that

$(\sum_{k=0}^{n}|A_k|)^t \le \sum_{k=0}^{n}|A_k|^t$

And

$E[\log \sum_{k=0}^{n}|A_k|] \le \frac{1}{t} E\,[\log(\sum_{k=0}^{n}|A_k|^t)]$.

Jensen's inequality and linearity of expectation now give that

$E[\log \sum_{k=0}^{n} |A_k|] \leq \frac{1}{t} \log E[\sum_{k=0}^{n} |A_k|^t] = \frac{1}{t} \log(\sum_{k=0}^{n} E[|A_k|^t])$.

Proof of Theorem 2.1. Observe that the leading coefficient of P_n is $A_n c_n$. Since $E[\log A_n] > \infty$, the probability that $A_n = 0$ is zero, and P_n is a polynomial of exact degree n with probability one. It gives the following estimate:

$$|(\tau_n - \mu_E)(S(I))| \leq 8\sqrt{\frac{1}{n} \log \frac{\|P_n\|_E}{|A_n c_n| (cap(E))^n}} \quad (4.9)$$

Using this estimate and Jensen's inequality, we obtain that

$$E[|(\tau_n - \mu_E)(S(I))|] \leq 8\sqrt{\frac{1}{n}\left(E[\log \|p_n\|_E] - \log\left(c_n (cap(E))^n\right) - E[\log |A_n|]\right)}.$$

It is clear that

$\|p_n\|_E \leq \sum_{k=0}^{n} |A_k| \, \|P_k\|_E \leq \max_{0 \leq k \leq n} \|P_k\|_E \sum_{k=0}^{n} |A_k|$.

Hence, (4.8) yields

$E[\log \|p_n\|_E] \leq E[\log \sum_{k=0}^{n} |A_k|] + \log \max_{0 \leq k \leq n} \|P_k\|_E \leq \frac{1}{t} \log(\sum_{k=0}^{n} E[|A_k|^t]) + \log \max_{0 \leq k \leq n} \|P_k\|_E$.

The leading coefficient c_n of the orthonormal polynomial p_n provides the solution of the following extremal problem:

$|c_n|^{-2} = \inf\{\int_E |Q_n(x)|^2 w(x)dx : Q_n \text{ is a monic polynomial of degree n}\}$.

We use a monic polynomial $Q_n(z)$ that satisfies $\|Q_n\|_E \leq C(cap(E)^n)$, where C > 0 depends only on E. Existence of such polynomial for a set E composed of finitely many smooth arcs and curves was first proved by Widom. Hence we estimate that

$c_n \geq \left(\int_E |Q_n(x)|^2 w(x)dx\right)^{-1/2} \geq \left(\int_E w(x)dx\right)^{-1/2} \|Q_n\|_E^{-1} \geq C^{-1}\left(\int_E w(x)dx\right)^{-1/2} \left(cap(E)\right)^{-1}$.

It follows that

$c_n \left(cap(E)\right)^n \geq C^{-1}\left(\int_E w(x)dx\right)^{-1/2}$, Where C depends only on E. Thus (2.1) follows by combining the above estimates.

Proof of Corollary 2.2. We estimate the right hand side of (2.1). For this purpose, we make two immediate observations that (2.3) implies

$\frac{1}{tn}\log(\sum_{k=0}^{n} E[|A_k|^t]) = O\left(\frac{\log n}{n}\right)$ as n → ∞.

While (2.4) implies

$-\frac{1}{n}\, E[\log|A_n|] \leq O\left(\frac{1}{n}\right)$ as n → ∞.

If (1.4) is satisfied, then

$\lim_{n\to\infty}\sup(\max_{0\leq k\leq n}\|p_k\|_E)^{1/n} = \lim_{n\to\infty}\sup(\|p_k\|_E)^{1/n} \leq 1,$

Which implies that

$\lim_{n\to\infty}\sup\frac{1}{n}\log\max_{0\leq k\leq n}\|p_k\|_E \leq 0.$

Hence, (2.5) follows from the above inequalities and (2.1). On the other hand, if (2.2) is satisfied, then

$\frac{1}{n}\log\max_{0\leq k\leq n}\|p_k\|_E = O\left(\frac{\log n}{n}\right)$ as n → ∞,

And (2.6) follows in the same manner.

Proof of Theorem 2.3. Let $Q_n(z) = \prod_{k=1}^{n}(z - z_k)$ be an arbitrary monic polynomial of degree n with the zero counting measure $\tau_n = \frac{1}{n}\sum_{k=1}^{n}\delta_{z_k}$. Since τ_E is a unit measure supported on E, we have that

$\int \log|Q_n(z)|\, d\mu_E(z) \leq \log\|Q_n\|_E.$

We need the well known representation of the Green function $g_E(z)$ for the complement of E with logarithmic pole at infinity:

$g_E(z) = \int \log|z - t|\, d\mu_E(t) - \log cap(E).$

Recall that $g_E(z) \geq 0$, z ∈ ℂ, and that $g_E(z)$ is a positive harmonic function in $\mathbb{C}/E$. Using Fubini's theorm and the above identity, we obtain that

$\frac{1}{n}\int \log|Q_n(z)|\,d\mu_E(z) = \iint \log|z-t|\,d\tau_n(t)d\mu_E(z) = \iint \log|z-t|\,d\mu_E(z)d\tau_n(t) = \int g_E(t)d\tau_n(t) + \log cap(E) = \frac{1}{n}\sum_{k=1}^{n} g_E(z_k) + \log cap(E) \geq \tau_n(S)\min_{z\in S} g_E(z) + \log cap(E)$.

Denoting b:= $min_S g_E$ > 0, we arrive at the inequality

b$\tau_n(S) + \log cap(E) \leq \frac{1}{n}\log\|Q_n\|_E$.

If we set $Q_n(z) = P_n(z)/(A_n c_n)$ and take the expectation, then the estimate becomes

E$[\tau_n(S)] \leq \frac{1}{bn}\left(E[\log\|P_n\|_E] - \log c_n\left(cap(E)\right)^n - E[\log|A_n|]\right)$.

The rest of this proof is identical to that of Theorem 2.1.

Proof of Corollary 2.4. We follow the same lines as in the proof of Corollary 2.2, but using (2.7) instead of (2.1). Namely, we again obtain from (2.2). (2.3) and (2.4) that

$\frac{1}{n}\log \max_{0\leq k\leq n}\|p_k\|_E$ = O $\left(\frac{\log n}{n}\right)$ as n → ∞,

$\frac{1}{tn}\log(\sum_{k=0}^{n} E[|A_k|^t])$ = O $\left(\frac{\log n}{n}\right)$ as n → ∞,

And

$-\frac{1}{n}\ E[\log|A_n|] \leq$ O$\left(\frac{1}{n}\right)$ as n → ∞.

Hence, (2.8) and (2.9) are immediate from (2.7).

4.3. Proofs for Section 3

Proof of Theorem 3.1. Recall that the leading coefficient of P_n^* given by $A_n c_n a_n{}^n$ does not vanish almost surely. Indeed, assumption E$[\log|A_n|] > -\infty$ gives the probability of A_n = 0 is zero, and c_n > 0. Hence, P_n^*has exactly n zeros with probability one denoted by $\{z_k\}_{k=1}^{n}$. We project the zeros of P_n^* onto the real line to construct the monic polynomial

M_n(z):= $\prod_{k=1}^{n}(z-\xi_k)$, ξ_k = Re z_k, 1 ≤ k ≤ n.

It is obvious that the zero counting measures $\nu_n = \frac{1}{n}\sum_{k=1}^{n}\xi_k$ for $M_n$satisfy $\nu_n\left(S(I)\right) = \tau_n{}^*\left(S(I)\right)$.

Note also that

$|M_n(x)| \le \frac{|P_n^*(x)|}{|A_n|c_n a_n^n}$, x ∈ ℝ. (4.10)

We use the discrepancy estimate for the signed measure $\sigma = \mu_\lambda - \nu_n$ on a sufficiently large interval L □ ℝ such that $[-1,1]$ and $\{\xi_k\}_{k=1}^n$. In the notation of that result, we have that $\sigma^+ = \mu_\lambda$ and $\sigma^+ = \nu_n$ are positive unit measures supported on L. We need to verify that the inequality $\mu_\lambda(I) \le C_{\mu[-1,1]}(I)$holds for all intervals I □ L with a fixed constant C > 0, where $d_{\mu[-1,1]} = dx/(\pi\sqrt{1-x^2})$ is the equilibrium measure of $[-1,1]$. This inequality is satisfied because both μ_λ and $\mu[-1,1]$ are supported on $[-1,1]$, the density of μ_λ is uniformly bounded above for λ > 1, and the density of $\mu[-1,1]$ is uniformly bounded below by a positive constant. We obtain that

$|(\tau_n^* - \mu_\lambda)(S(I))| = |(\mu_\lambda - \nu_n)(S(I))| \le D[\sigma] \le C_1\sqrt{\varepsilon}$, (4.11)

Where C_1> 0, and ε is expressed through the logarithmic potential of σ denoted by U^σ.

$$\varepsilon = \sup_{z\in\mathbb{C}} U^\sigma(z) = \sup_{z\in\mathbb{C}}(U^{\mu_\lambda}(z) - U^{\nu_n}(z)) = \sup_{z\in\mathbb{C}}\left(U^{\mu_\lambda}(z) + \frac{1}{n}\log|M_n(z)|\right)$$

Since the function $U^{\mu_\lambda}(z) + \frac{1}{n}\log|M_n(z)|$ is subharmonic in $\bar{\mathbb{C}} \setminus [-1,1]$, we obtain that

$$\varepsilon = \sup_{x\in[-1,1]}\left(U^{\mu_\lambda}(x) + \frac{1}{n}\log|M_n(x)|\right).$$

Recall that μ_λ is the weighted equilibrium measure corresponding to the weight w_λ(x) = $e^{-\gamma_\lambda|x|^\lambda}$on ℝ. This implies that

$U^{\mu_\lambda}(x) = F_\lambda - \gamma_\lambda|x|^\lambda$, x ∈ $[-1,1]$,

Where $F_\lambda = \log 2 + 1/\lambda$ is the modified Robin constant corresponding to w_λ. Hence, we obtain from the above and (4.10) that

$$\varepsilon = \sup_{x\in[-1,1]}\left(F_\lambda - \gamma_\lambda|x|^\lambda + \frac{1}{n}\log|M_n(x)|\right) \le \sup_{x\in[-1,1]}\left(F_\lambda - \gamma_\lambda|x|^\lambda + \frac{1}{n}\log\frac{|P_n^*(x)|}{|A_n|c_n a_n^n}\right)$$

$$= \sup_{x\in[-1,1]}\left(F_\lambda + \frac{1}{n}\log\frac{w_\lambda^n|P_n^*(x)|}{|A_n|c_n a_n^n}\right) = F_\lambda + \frac{1}{n}\log\|w_\lambda^n P_n^*\|_{[-1,1]} - \frac{1}{n}\log(|A_n|c_n a_n^n).$$

Combining this estimate with (4.11), we arrive at

$$|(\tau_n^* - \mu_\lambda)(S(I))| \le C_1\sqrt{\log 2 + 1/\lambda + \frac{1}{n}\log\|w_\lambda{}^n P_n{}^*\|_{[-1,1]} - \frac{1}{n}\log(|A_n|c_n a_n{}^n)}$$

Jensen's inequality now gives that

$$\mathrm{E}[|(\tau_n^* - \mu_\lambda)(S(I))|] \le C_1\sqrt{\frac{1}{n}\left(E\left[\log\|w_\lambda{}^n P_n{}^*\|_{[-1,1]}\right] - E[\log|A_n|] + \log\frac{2^n e^{n/\lambda}}{c_n a_n{}^n}\right)}. \qquad (4.12)$$

It follows that

$$\lim_{n\to\infty}\frac{c_n\sqrt{\pi}}{2^n e^{n/\lambda}}\left(\frac{n\gamma_\lambda}{c}\right)^{(n+1/2)/\lambda} = 1.$$

Since $a_n = \gamma_\lambda^{\frac{1}{\lambda}} c^{\frac{-1}{\lambda}} n^{\frac{1}{\lambda}}$, we obtain that

$$\frac{2^n e^{n/\lambda}}{c_n a_n{}^n} = \frac{2^n}{c_n}\left(\frac{ec}{n\gamma_\lambda}\right)^{n/\lambda} = \mathrm{O}\left(n^{\lambda/2}\right),$$

where the constant in the O-term depends only on c and λ.

It remains to estimate the term $\mathrm{E}\left[\log\|w_\lambda{}^n\|_{[-1,1]}\right]$ in (4.12).

We obtain that

$$\|w_\lambda{}^n P_n{}^*\|_{[-1,1]} = \|w_\lambda{}^n P_n{}^*\|_{\mathbb{R}} = \|P_n W\|_{\mathbb{R}} \le \sum_{k=0}^{n}|A_k|\|p_k W\|_{\mathbb{R}} \le \max_{0\le k\le n}\|p_k W\|_{\mathbb{R}}\sum_{k=0}^{n}|A_k|.$$

Hence, (4.8) gives

$$\mathrm{E}\left[\log\|w_\lambda{}^n P_n{}^*\|_{[-1,1]}\right] \le \mathrm{E}[\log\sum_{k=0}^{n}|A_k|] + \log\max_{0\le k\le n}\|p_k W\|_{\mathbb{R}}$$

$$\le \frac{1}{t}\log(\sum_{k=0}^{n} E[|A_k|^t]) + \log\max_{0\le k\le n}\|p_k W\|_{\mathbb{R}}.$$

Applying the Nikolskii-type inequality, we estimate

$$\|p_k W\|_{\mathbb{R}} \le \mathrm{O}\left(k^{\frac{\lambda-1}{2\lambda}}\right)\left(\int_{-\infty}^{\infty}|p_k(x)|^2 W^2(x)dx\right)^{1/2} \le \mathrm{O}\left(n^{\frac{\lambda-1}{2\lambda}}\right),\ 0 \le k \le n,$$

where we used that the polynomials p_k are orthonormal with respect to W on ℝ. Thus (3.1) follows from (4.12) and the above estimates.

Proof of Corollary 3.2. Equation (3.4) follows from (3.1) by essentially the same argument as in the proof of Corollary 2.2, where we deduce (2.6) from (2.1).

ACKNOWLEDGEMENTS

Results of this paper were presented at the Ⅶ Jaen Conference on Approximation held in Úbeda, July 3-8, 2016. The author would like to thank the organizers for the invitation and hospitality.

REFERENCES

1. Bayraktar T. (2016) Equidistribution of zeros of random holomorphic sections, Indiana Univ. Math. J., 65(5): 17591793.
2. Bloom T. and N. Levenberg (2015) Random polynomials and pluripotential-theoretic extremal functions, Potential Anal., 42(2): $311-334$.
3. Ibragimov I. and D. Zaporozhets (2013) On distribution of zeros of random polynomials in complex plane, Prokhorov and contemporary probability theory, Springer Proc. Math. Stat., 33: 303-323.
4. Kabluchko Z. and D. Zaporozhets (2013) Roots of random polynomials whose coefficients have logarithmic tails, Ann. Probab, 41(5): $3542-3581$.
5. Kabluchko Z. and D. Zaporozhets (2014) Asymptotic distribution of complex zeros of random analytic functions, Ann. Probab, 42(4): $1374-1395$.
6. Lubinsky D. S., I. E. Pritsker and X. Xie (2016) Expected number of real zeros for random linear combinations of orthogonal polynomials, Proc. Amer. Math. Soc., 144(4): 1631-1642.
7. Pritsker I. E. (2016) Asymptotic zero distribution of random polynomials spanned by general bases, Contemp. Math, $661,121-140$.
8. Pritsker I. E. and X. Xie (2015) Expected number of real zeros for random Freud orthogonal polynomials, J. Math. Anal. Appl., $429(2),1258-1270$.
9. Shiffman B. and S. Zelditch (2011) Random complex fewnomials, I, in: P. Brändén, M. Passare and M. Putinar (eds.), Notions of Positivity and the Geometry of Polynomials, Trends in Mathematics, pp. 375-400, Birkhäuser / Springer Basel AG, Basel.