wjert, 2025, Vol. 11 Issue 8, 68-77 Review Article ISSN 2454-695X

World Journal of Engineering Research and Technology
WJERT
www.wjert.org
SJIF Impact Factor: 7.029

GEOMETRIC CONDITIONS FOR HOMOTHETIC-TO-PROJECTIVE CONVERSIONS IN FIFTH-ORDER RECURRENT BY LIE-DERIVATIVE IN FINSLER SPACES

Adel M. Al-Qashbari[1], Alaa A. Abdallah*[2] and Saeedah M. Baleedi[3]

[1]Department of Mathematics, Education Faculty- Aden, University of Aden, Aden, Yemen.

[1]Department of Engineering, Faculty of the Engineering and Computers, University of Science and Technology, Aden, Yemen.

[2*]Department of Mathematics, Dr. Babasaheb Ambedkar Marathwada University, Aurangabad, India.

[2,3]Deptment of Mathematics, Education Faculty- Zingibar, University of Abyan, Zingibar, Yemen.

Article Received on 11/06/2025 **Article Revised on 01/07/2025** **Article Accepted on 22/07/2025**

***Corresponding Author**
Dr. Alaa A. Abdallah
Department of Mathematics, Dr. Babasaheb Ambedkar Marathwada University, Aurangabad, India.

ABSTRACT

In this paper, we introduce five conditions for converting certain types of homothetic motions into projective motions in the generalized fifth recurrent Finsler space by Lie-derivative. Further prove that the Lie-derivative of the Cartan's fourth curvature tensor K^i_{jkh} and *h(v)*-torsion tensor H^i_{kh} behave as fifth recurrent within this space that allows the conversion.

KEYWORDS: Projective motion, Homothetic motion, $\mathcal{GBK}-5RF_n$, Berwald connection, Lie-derivative.

1. INTRODUCTION

A homothetic motion is credited to the German mathematician Gottlob Friedrich Helmholtz, who first studied the homothetic motion in (1857). The word “Hemothetice” is derived from the Greek word “Hemothesis” meaning “relative position”, it is used to describe geometric transformations that preserve size and direction. Hiramatu[10] established a homothetic transformation in a Finsler space. Singh[14] proved that every homothetic transformation in a

Finsler space is a W −curvature collineation. Opondo[12] discussed the affine and homothetic motions in bi-recurrent Finsler space.

The projective changes of Finsler metrics and projectively flat Finsler space introduced by Matsumoto.[11] Several theorems in Projective motion proved by Verma.[15] Al-Qashbari et al.[3-6] studied projective motion in generalized fifth recurrent Finsler space by Lie-derivative and established various identities on Lie-derivative of tensors within this space. Pandey[13] studied Lie-derivative in recurrent Finsler space.

Several relations on curvature tensor of different order in Finsler space, and important results of generalized Finsler space in sense of Berwald established by Abdallah and et al.[1,2,8,9] The necessary and sufficient condition for some tensors in generalized recurrent Finsler space introduced by.[7] The aim of this paper is to study conversion a body's homothetic motion (around its own axis) into projective motion along its curved path (around gravitational source) in generalized fifth recurrent Finsler space by Lie-derivative. This allows for precisely determine the body's actual position in the projective path.

2. Preliminaries

A Lie - derivative evaluate the rate of change of a vector field or a tensor field along the smooth vector field $v^i(x)$. The Lie-derivative of a general mixed tensor field T^i_{jkh} is given by following formula

(2.1) $L_v T^i_{jkh} = v^m \mathcal{B}_m T^i_{jkh} - T^m_{jkh} \mathcal{B}_m v^i + T^i_{mkh} \mathcal{B}_j v^m + T^i_{jmh} \mathcal{B}_k v^m$

$$+ T^i_{jkm} \mathcal{B}_h v^m + \dot{\partial}_m T^i_{jkh} \mathcal{B}_r v^m y^r,$$

Where $v^m \neq 0$ is a contravariant vector field independent of directional argument and dependent on positional coordinates (x^i) only and $\mathcal{B}_j v^m = 0$.

The Lie-derivative of the metric tensor g_{ij} is vanishing, i.e.

(2.2) $L_v g_{ij} = 0$.

The projective motion it is the motion of an object in curved space that is not effected by any forces other than gravity and moves along curved paths .the sufficient condition for motion to become projective is[15]

(2.3) $L_v G^i_{kh} = \delta^i_k P_h + \delta^i_h P_k + y^i P_{kh}$.

The homothetic motion it is the motion of an object in curved space where the object's angular velocity is constant at any point along its curved path. the sufficient condition for motion to become homothetic is[12]:

(2.4) $L_v g_{ij} = 2c g_{ij}$ (where c is a constant).

The Cartan's fourth curvature tensor K^i_{jkh}, Cartan's third curvature tensor R^i_{jkh} and $h(v)$ −torsion tensor H^i_{kh} satisfy the following relation

(2.5) $K^i_{jkh}\, y^j = R^i_{jkh}\, y^j = H^i_{kh}$.

The Ricci tensor P_{jk} and non-zero vector y^k satisfies the following relation

(2.6) $P_{jk} y^k = 0$.

The metric tensor g_{ij} and Kronecker delta δ^i_h are satisfying the relation

(2.7) $g_{ij}\, g^{ik} = \delta^k_j = \begin{cases} 1 & if\ \ j = k \ , \\ 0 & if\ \ j \neq k \ . \end{cases}$

In generalized fifth recurrent Finsler space for K^i_{jkh} in sense of Berwald $(GBK - 5RF_n)$, some projective motions are given by[3]

(2.8) $L_v(\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m K^i_{jkh}) = K^i_{jkh}\, v^m\, \mathcal{B}_m a_{sqlnm} + a_{sqlnm} v^m\, \mathcal{B}_m K^i_{jkh} + y^i P_{kh}$.

(2.9) $L_v(\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m H^i_{kh}) = H^i_{kh}\, v^m\, \mathcal{B}_m a_{sqlnm} + a_{sqlnm} v^m\, \mathcal{B}_m H^i_{kh} + y^j y^i P_{kh}$.

(2.10) $P_{kh} = F^2 v^m\, \mathcal{B}_m P_{kh}$.

(2.11) $\mathcal{B}_m b_{sqlnm} = -\frac{y^j y^k P_{kh}}{y_h v^m}$.

(2.12) $L_v(\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m R^i_{jkh}) = L_v(\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m K^i_{jkh})$,

if

(2.13) $L_v\left(a_{sqlnm} R^i_{jkh}\right) = L_v(a_{sqlnm} K^i_{jkh})$.

The Lie-derivative of the vector field b_{sqlnm} of fifth order vanishes identically, i.e.

(2.14) $L_v b_{sqlnm} = 0$.

The Berwald connection parameter G^i_{kh} vanishins in $GBK - 5RF_n$, i.e.

(2.15) $G^i_{kh} = 0$.

The Lie-derivative and by Berwald's covariant derivative of fifth order for Cartan's fourth curvature tensor K^i_{jkh} are commutative, if the Lie-derivative of non-zero vector field of fifth order a_{sqlnm} vanishes simultaneously with the vanishing of scalar function $\alpha(x)$ by Berwald's covariant derivative of fifth order, i.e.

(2.16) $\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m(L_v K^i_{jkh}) = L_v(\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m K^i_{jkh})$,

if

(2.17) $L_v a_{sqlnm} = \mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m \alpha(x) = 0$.

The Cartan's fourth curvature tensor K^i_{jkh} and the h(v)-torsion tensor H^i_{kh} within $G\mathcal{B}K-5RF_n$, are satisfying the relations

(2.18) $\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m K^i_{jkh} = a_{sqlnm} K^i_{jkh} + b_{sqlnm}(\delta^i_h g_{jk} - \delta^i_k g_{jh})$
$-c_{sqlnm}(\delta^i_h C_{jkn} - \delta^i_k C_{jhn}) - d_{sqlnm}(\delta^i_h C_{jkl} - \delta^i_k C_{jhl})$
$-e_{sqlnm}(\delta^i_h C_{jkq} - \delta^i_k C_{jhq}) - 2b_{qlnm} y^r \mathcal{B}_r(\delta^i_h C_{jks} - \delta^i_k C_{jhs})$

and

(2.19) $\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m H^i_{kh} = a_{sqlnm} H^i_{kh} + b_{sqlnm}(\delta^i_h y_k - \delta^i_k y_h)$.

The Lie-derivative and Berwald's covariant derivative of the vector y^j are vanishing identically, i.e.

(2.20) a) $L_v y^j = 0$ and b) $\mathcal{B}_m y^j = 0$.

3. Main Results

In this section, we established five conditions for conversion five certain Homothetic motions to projective motions in generalized fifth recurrent Finsler space by Lie-derivative.

Condition (1)

Theorem 3.1: *In $G\mathcal{B}K$-$5RF_n$, the condition (3.2) converts the homothetic motion (3.3) into the projective motion (2.8).*

Proof: Using (2.7) and (2.15) in (2.3), then since $y^i \neq 0$, we get

(3.1) $P_{kh} = 0$.

Transvecting above equation by y^i and using (2.2) in the right side of result equation, we get

(3.2) $y^i P_{kh} = L_v g_{kh}$.

Using (2.4) in above equation, we get

(3.3) $y^i P_{kh} = 2c g_{kh}$.

Above equation represents a homothetic motion. Now using (2.4) and (2.2) in right side of above equation, then multiplying the result equation by (-1), we get

$$-y^i P_{kh} = 0.$$

Which can be written as

$$-y^i P_{kh} = L_v(a_{sqlnm} K^i_{jkh}) - L_v(a_{sqlnm} K^i_{jkh}).$$

Or

$$L_v(a_{sqlnm} K^i_{jkh}) = L_v(a_{sqlnm} K^i_{jkh}) + y^i P_{kh}.$$

Using (2.7) in (2.18), then using the result equation in the left side of above equation, we get

(3.4) $L_v(\mathcal{B}_s \mathcal{B}_q \mathcal{B}_l \mathcal{B}_n \mathcal{B}_m K^i_{jkh}) = K^i_{jkh} L_v(a_{sqlnm}) + a_{sqlnm} L_v K^i_{jkh} + y^i P_{kh}.$

Applying formula (2.1) to the tensors a_{sqlnm} and K^i_{jkh}, then using the result equations in above equation, we obtained the projective motion (2.8).

Condition (2)

Theorem 3.2: *In* $GBK\text{-}5RF_n$, *the condition (3.5) converts the homothetic motion (3.6) into the projective motion (2.9).*

Proof: Transvecting (3.1) by $y^j y^i$, then using (2.2) in the right side of the result equation, we get

(3.5) $y^j y^i P_{kh} = L_v g_{kh}.$

Using (2.4) in above equation, we get

(3.6) $y^j y^i P_{kh} = 2c g_{kh}.$

Above equation represents a homothetic motion. Now using (2.4) and (2.2) in right side of above equation, then multiplying the result equation by (-1), we get

$- y^j y^i P_{kh} = 0.$

Which can be written as

$- y^j y^i P_{kh} = L_v(a_{sqlnm} H^i_{kh}) - L_v(a_{sqlnm} H^i_{kh}).$

Or

$L_v(a_{sqlnm} H^i_{kh}) = L_v(a_{sqlnm} H^i_{kh}) + y^j y^i P_{kh}.$

Using (2.7) in (2.19), then using the result equation in the left side of above equation, we get

(3.7) $L_v(\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m H^i_{kh}) = H^i_{kh} L_v(a_{sqlnm}) + a_{sqlnm} L_v H^i_{kh} + y^j y^i P_{kh}$.

Applying formula (2.1) to the tensors a_{sqlnm} and H^i_{kh}, then using the result equations in above equation, we obtained the projective motion (2.9).

Now, we have two corollaries related to the previous theorems. Using (2.16) and (3.1) in (3.4), we get

$\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m(L_v K^i_{jkh}) = a_{sqlnm}(L_v K^i_{jkh})$.

Above equation means the Lie-derivative of the Cartan's fourth curvature tensor K^i_{jkh} behaves as fifth recurrent. Thus, we conclude

Corollary 3.1: *In the main space that allows the conversion of homothetic motion (3.3) into the projective motion (2.8), the Lie-derivative of the Cartan's fourth curvature tensor* K^i_{jkh} *behaves as fifth recurrent [provided (2.17) holds].*

Using (2.16) and (3.1) in (3.7), we get

(3.8) $L_v(\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m H^i_{kh}) = a_{sqlnm}(L_v H^i_{kh})$.

Transvecting (2.16) by y^j, using [(2.20)a,b] and (2.5), we get

$$\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m(L_v H^i_{kh}) = L_v(\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m H^i_{kh}).$$

Using above equation in (3.8), we get

$$\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m(L_v H^i_{kh}) = a_{sqlnm}(L_v H^i_{kh}).$$

Above equation means the Lie-derivative of the $h(v)$ −torsion tensor H^i_{kh} behaves as fifth recurrent. Thus, we conclude

Corollary 3.2: *In the main space that allows the conversion of homothetic motion (3.6) into the projective motion (2.9), the Lie-derivative of the h(v)-torsion tensor* H^i_{kh} *behaves as fifth recurrent [provided (2.17) holds].*

Condition (3)

Theorem 3.3: *In* $GBK\text{-}5RF_n$, *the condition (3.9) converts the homothetic motion (3.10) into the projective motion (2.10).*

Proof: Taking the Lie-derivative of both sides of (3.1), then using (2.2) in the right side of the result equation, we get

$$L_v P_{kh} = L_v g_{kh}.$$

Transvecting above equation by the quadrature fundamental function of Finsler space F^2, we get

$$F^2 L_v P_{kh} = F^2 L_v g_{kh}.$$

Applying formula (2.1) to the P −Ricci tensor P_{kh}, then using the result equations in above equation, we

(3.9) $F^2 v^m \mathcal{B}_m P_{kh} = F^2 L_v g_{kh}$.

Using (2.4) in above equation, we get

(3.10) $F^2 v^m \mathcal{B}_m P_{kh} = 2c(F^2 g_{kh})$.

Above equation represents a homothetic motion. Now using (2.4), (2.2) and (3.1) in right side of above equation, we get

$$F^2\, v^m \mathcal{B}_m P_{kh} = P_{kh} - 2c g_{kh}.$$

Which can be written as

$$P_{kh} - F^2\, v^m \mathcal{B}_m P_{kh} = 2c g_{kh}.$$

Using (2.4) and (2.2) in above equation, we obtained the projective motion (2.10).

Condition (4)

Theorem 3.4: *In $G\mathcal{B}K\text{-}5RF_n$, the condition (3.11) converts the homothetic motion (3.12) into the projective motion (2.11).*

Proof: Transvecting (2.14) by y_h, then using (2.2) in right side of above equation, we get

(3.11) $y_h L_v b_{sqlnm} = L_v g_{kh}$.

Using (2.4) in above equation, we get

(3.12) $y_h L_v b_{sqlnm} = 2c g_{kh}$.

Above equation represents a homothetic motion. Now using (2.4) and (2.2) above equation, we get

$$y_h L_v b_{sqlnm} = 0 .$$

Transvecting (3.1) by ($-y^j y^k$), then using the result equation in above equation, we get

$$y_h L_v b_{sqlnm} = -y^j y^k P_{kh}.$$

Applying formula (2.1) to the vector field of fifth order b_{sqlnm}, then using the result equations in above equation, we obtained the projective motion (2.11).

Condition (5)

Theorem 3.5: *In $\mathcal{GBK}$-$5RF_n$, the condition (3.14) converts the homothetic motion (3.15) into the projective motion (2.12).*

Proof: In view of (2.5), since $y^j \neq 0$, we get

(3.13) $R^i_{jkh} = K^i_{jkh}$.

Using (2.2) in above equation, we get

(3.14) $R^i_{jkh} - K^i_{jkh} = L_v g_{kh}$.

Using (2.4) in above equation, we get

(3.15) $R^i_{jkh} - K^i_{jkh} = 2c g_{kh}$.

Above equation represents a homothetic motion. Now taking the Berwald covariant derivative of fifth order for above equation with respect to x^m, x^n, x^l, x^q and x^s respectively, and using (2.4) and (2.2), we get

(3.16) $\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m R^i_{jkh} = \mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m K^i_{jkh}$.

Using (2.7) in (2.18), we get

(3.17) $\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m K^i_{jkh} = a_{sqlnm} K^i_{jkh}$.

Transvecting (2.5) by a_{sqlnm}, we get

$$a_{sqlnm} K^i_{jkh} = a_{sqlnm} R^i_{jkh} .$$

Using (3.17) and above equation in (3.16), we get

(3.18) $\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m R^i_{jkh} = a_{sqlnm} R^i_{jkh}$.

Using (3.17) and (3.18) in condition (2.13), we obtained the projective motion (2.12).

4. CONCLUSIONS

Five conditions convert certain homothetic motions into projective motions in $\mathcal{GBK}\text{-}5RF_n$ by Lie-derivative have been discussed. Also, we obtained certain types of tensors with Lie-derivative that behave as fifth recurrent in the main space.

REFERENCES

1. Abdallah, A. A. and Hardan B.: P −third order generalized Finsler space in the Berwald sense, Bulletin of Pure and Applied Sciences Section-E-Mathematical & Statistics, 2024; 43E(1): 43-52.
2. Abdallah, A. A., Navlekar A. A., Ghadle K. P. and Hardan B.: Important results of P − generalized Finsler space with Berwald connection of third order, Journal of Mathematical Problems, Equations and Statis tics, 2025; 6(1): 19-27.
3. AL-Qashbari, A. M., Abdallah, A. A. and Baleedi, S. M.: Projective motions in generalized fifth recurrent Finsler space via Lie derivative of Berwald covariant tensors, International Journal of Research Publication and Reviews, 2025; 6(4): 16101-16106.
4. AL-Qashbari, A. M., Abdallah, A. A. and Baleedi, S. M.: Berwald covariant derivative and Lie derivative of conharmonic curvature tensors in generalized fifth recurrent Finsler space, GPH- International Journal of Mathematics, 2025; 8(1): 24-32.
5. AL-Qashbari, A. M. and Baleedi, S. M.: Study on generalized $\mathcal{BK} - 5$ recurrent Finsler space, Computational Mathematics and its Ap plications, 2023; 1(1): 009-020.
6. AL-Qashbari, A. M. and Baleedi, S. M.: On Lie-derivative of M−projective curvature tensor and K−curvature inheritance in GBK− 5RFn, Acta. Universities Apuleius, 2023; 76: 13-24.
7. Ghadle, K. P., Navlekar, A. A. and Abdallah, A. A.: The necessary and sufficient condition for some tensors which satisfy a generalized $\mathcal{BP}$ −recurrent Finsler space, International Journal of Scientific and Engineering Research, 2019; 10(11): 135-140.
8. Ghadle, K. P., Navlekar, A. A., Abdallah, A. A. and Hardan, B.: On W^i_{jkh} generalized BP-recurrent and birecurrent Finsler space, AIP Confer ence Proceedings, 2024; 3087: 070001 (1-6).
9. Hamoud, A. A., Navlekar, A. A., Abdallah, A. A. and Ghadle, K. P.: Decomposition for Cartan's second curvature tensor of different order in Finsler spaces, Nonlinear Functional Analysis and Applications, 2022; 27(2): 433-448.

10. Hiramatu, H.: Groups of homothetic transformation in a Finsler space, Tensor New Series, 1954; 3: 131-143.
11. Matsumoto, M.: Projective changes of Finsler metrics and projectively flat Finsler spaces, Tensor N.S., 1980; 34: 303-3015.
12. Opondo, M. A.: *Study of projective curvature tensor* W^i_{jkh} *in bi-recurrent Finsler space,* Ph.D. Dissertation, Kenyatta University, (Nairobi), (Kenya), 2021.
13. Pandey, P. N.: On Lie-recurrent Finsler manifolds, Differential Geometry and Dynamical Systems, 2011; 13: 201-207.
14. Singh, S. P.: W −curvature collineation in Finsler space, Tensor New Series, 2013; 74(3): 252-257.
15. Verma, R.: *Some transformations in Finsler spaces*, D. Phil. Thesis, University of Allahabad, (Allahabad) (India), 1991.