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SOLUTION OF TWO DIMENSIONAL BURGERS EQUATION BY USING HYBRID CRANK-NICHOLSON AND LAX-FREDRICHS' FINITE DIFFERENCE SCHEMES ARISING FROM OPERATOR SPLITTING

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ABSTRACT

Solving Burgers equation continues to be a challenging problem. Burgers' equation is a fundamental partial differential equation from fluid mechanics. It occurs in various areas of applied mathematics, such as modeling of gas dynamics and traffic flow. It relates to the Navier-Stokes equation for incompressible flow with the pressure term removed. So far the methods that have been used to solve such equations are: Alternative Direction Implicit (ADI) methods, Variation of Iteration Method (VIM), locally one dimensional method and Finite Difference Method (FDM) approach which is used in this work. In this paper the pure Crank-Nicholson (CN) Scheme and Crank-Nicholson-Lax-Fredrichs' (CN-LF) method is developed by Operator Splitting. Crank-Nicholson-Du-Fort and Frankel is an hybrid scheme made by combining the Crank-Nicholson and Lax-Fredrich schemes. Lax-Friedrichs' scheme is conditionally stable and an explicit scheme. The developed schemes are solved numerically using initially solved solution via Hopf-Cole transformation and separation of variables to generate the initial and boundary conditions. Analysis of the resulting schemes was found to be unconditionally stable. The results of the hybrid scheme are found to compare well with those of the pure Crank-Nicholson.

KEYWORDS: Burgers-Equation, Operator-Splitting, Finite-Difference-Methods (FDM), Crank- Nicholson.

1. INTRODUCTION

Burgers' equations occur very frequently in science, engineering and mathematics. Many partial differential equations cannot be solved by analytical methods in closed form solution. In most research work in fields like: applied elasticity, theory of plate and shells, hydro-dynamics, quantum mechanics among others, the research problems reduce to partial differential equations. Various Numerical approaches to solve the Burgers' equations have been used in the past. Certain types of boundary value problems can be solved by replacing the differential equation by the corresponding finite difference equation and then solving the latter by a process of iteration. These methods have been used by many mathematicians according to Jain [2004]. Linearized parabolic equations appear as models in heat flow and gas dynamics. Finite difference solutions of these equations are found by using ordinary discretization (see (Ames, 1994) and Mitchell and Grffiths [1980]). These methods give fairly accurate results.

The Burgers' equation was first introduced by Bateman (1915) and studied in details by (Burgers, A Mathematical Model illustrating the theory of turbulence, 1948). Analytic solution of the Burgers' equation involves series solutions which converge very slowly for small values of viscousity constant according to Idris (2007).

(Espen, 2011) discussed numerical quadratures in one and two dimensions, which was followed by a discussion regarding the differentiation of general operators in Banach spaces. In the research they investigated the Godunov and Strang method numerically for the viscous Burgers' equation and the KdV equation and presented different numerical methods for the subequations from the splitting. They discovered that the Operator splitting methods work well numerically for the two equations. (Chang, Improved alternating-direction implicit method for solving transient three-dimensional heat diffusion problems, 1991).

2. DEVELOPMENT OF THE HYBRID SCHEMES

The 2-D Burgers equation is of the form:

$$\left.\begin{aligned} u_t + uu_x + vu_y &= \frac{1}{R}(u_{xx} + u_{yy}) \\ v_t + uv_x + vv_y &= \frac{1}{R}(v_{xx} + v_{yy}) \end{aligned}\right\} \quad (2.1)$$

Subject to initial conditions:

$$\left.\begin{aligned} u(x,y,0) &= f(x,y),\ \ (x,y) \in D \\ v(x,y,0) &= g(x,y),\ \ (x,y) \in D \end{aligned}\right\} \quad (2.2)$$

and boundary conditions:

$$\left.\begin{aligned} u(x,y,t) &= f_1(x,y),\ \ x,y \in \partial D, t > 0 \\ v(x,y,t) &= g_1(x,y),\ \ x,y \in \partial D, t > 0 \end{aligned}\right\} \quad (2.3)$$

Where $D = \{(x,y) | a \le x \le b, a \le y \le b\}$ and ∂D is its boundary $u(x,y,t)$ and $v(x,y,t)$ are the velocity components to be determined, f, g, f_1 and g_1 are known functions and R is the Reynolds number.

Which is a fundamental partial differential equation in fluid mechanics and it occurs in various areas of applied mathematics, such as modeling of gas dynamics, heat conduction, and acoustic waves (Hongqing [2010]).

2.1 Overview of Operator Splitting

Consider the Taylor's expansion

$$\begin{aligned} u(x,y,t+k) &= u(x,y,t) + k\frac{\partial}{\partial t}u(x,y,t) + \frac{k^2}{2!}\frac{\partial^2}{\partial t^2}u(x,y,t) + \cdots \\ &= \left(1 + k\frac{\partial}{\partial t} + \frac{k^2}{2!}\frac{\partial^2}{\partial t^2} + \cdots\right) u(x,y,t) \\ &= e^{k\frac{\partial}{\partial t}}\left(u(x,y,t)\right) \end{aligned} \quad (2.4)$$

In equation (2.4) we can replace $\frac{\partial}{\partial t}$ by L that is

$$u(x,y,t+k) = e^{kL}\, u(x,y,t) \quad (2.5)$$

The exact solution of the equation (2.1) at the grid point $(x = mh, y = lq, t = nk)$ is $u(x,y,t)$ with h, q and k being the grid spacing in the x- direction, y- direction and t-direction respectively. m, l and k are intergers. $m = l = n = 0$ is the origin. The approximate solution at this point is denoted by $U_{m,l,n}$. The finite difference (FD) approximation of equation (2.5) can thus be expressed as:

$$U_{m,l,n+1} = e^{kL}\, U_{m,l,n} \quad (2.6)$$

In equations (2.5) and (2.6) e^{kL} is called the solution operator for equation (2.1) L is replaced by finite difference approximation. In equation (2.6) L can be taken to be a sum of differential operators with respect to x.

If $L = L_1 + L_2 + L_3 + \cdots + L_S = \sum_{i=1}^{S} L_i$

Then equation (2.6) can be written as

$$U_{m,l,n+1} = e^{k\sum_{i=1}^{s} L_i}\, U_{m,l,n}$$

$$= e^{k\,(L_1 + L_2 + L_3 + \cdots + L_S)}\, U_{m,l,n} \qquad (2.7)$$

$$= (e^{kL_1}\,(e^{kL_2}(\ldots.(e^{kL_{s-1}}(e^{kL_s}U_{m,l,n}\ ))\ \ldots.))) \qquad (2.8)$$

$$= \prod_{i=1}^{s} e^{kL_i}\, U_{m,l,n} \qquad (2.9)$$

The approximate solution can be obtained from equation (2.8) by first solving

$$U_{m,l,n+1}^{(s)} = e^{kL_s}U_{m,l,n} \qquad (2.10)$$

and then using this solution we can find

$$U_{m,l,n+1}^{(s-1)} = e^{kL_{s-1}}U_{m,l,n} \qquad (2.11)$$

We go on like this until we attain

$$U_{m,l,n+1}^{(1)} \qquad (2.12)$$

which is actually the approximate solution of equation (2.1)

2.2 Pure Crank-Nicholson (CN) Scheme

We consider the 2-D Burgers equation of the form

$$f_1(u,v) = -uu_x - vu_y + \frac{1}{Re}\left(u_{xx} + u_{yy}\right), (0 \le x, y \le 1) \times (0 \le t \le T) \qquad (2.13)$$

$$f_2(u,v) = -uv_x - vv_y + \frac{1}{Re}\left(v_{xx} + v_{yy}\right), (0 \le x, y \le 1) \times (0 \le t \le T) \qquad (2.14)$$

Here s=2

And so

$L = L_1 + L_2 + L_3 + L_4$

Let

$$L_1 = -\frac{1}{h} u_{m,l,n}\delta_x$$

$$L_2 = -\frac{1}{q} v_{m,l,n}\delta_y$$

$$L_3 = \frac{1}{Re\,h^2}{\delta_x}^2$$

$$L_4 = \frac{1}{Re\,q^2}{\delta_y}^2$$

From equation (2.8) the approximate solution is found by

$$U_{m,l,n+1} = (e^{kL_1}(e^{kL_2}(e^{kL_3}(e^{kL_4}U_{m,l,n})))) \quad (2.15)$$

$$U_{m,l,n+1} = \left(1+kL_1+\tfrac{1}{2}k^2L^2{}_1+..\right)\left(1+kL_2+\tfrac{1}{2}k^2L^2{}_2+..\right)\left(1+kL_3+\tfrac{1}{2}k^2L^2{}_3+..\right)\left(1+kL_4+\tfrac{1}{2}k^2L^2{}_4+..\right)$$

$$\approx 1 + kL_1 + kL_2 + kL_3 + kL_4 + L_1L_2k^2 + L_1L_3k^2 + L_1L_4k^2 + L_2L_3k^2 + L_2L_4k^2 + L_3L_4k^2 + \tfrac{1}{2}kL_1{}^2 + \tfrac{1}{2}kL_2{}^2 + \tfrac{1}{2}kL_3{}^2 + \tfrac{1}{2}kL_4{}^2 \quad (2.16)$$

It is necessary that we first develop the pure Crank-Nicholson method resulting from this splitting. This is because other hybrid methods are derived from it. Thus the Crank-Nicholson method is as follows:

$$L_1 U_{m,l,n} = -\frac{1}{2h}U_{m,l,n}\delta_x(U_{m,l,n} + U_{m,l,n+1}) \quad (2.17)$$

$$L_2 V_{m,l,n} = -\frac{1}{2q}V_{m,l,n}\delta_y(U_{m,l,n} + U_{m,l,n+1}) \quad (2.18)$$

$$L_3 U_{m,l,n} = \frac{1}{4Re\,h^2}\delta_x^2(U_{m,l,n} + U_{m,l,n+1}) \quad (2.19)$$

$$L_4 U_{m,l,n} = \frac{1}{4Re\,q^2}\delta_y^2(U_{m,l,n} + U_{m,l,n+1}) \quad (2.20)$$

$$L_1L_2V_{m,l,n} = \frac{1}{4hq}U_{m,l,n}V_{m,l,n}\,\delta_x(\delta_y(U_{m,l,n} + U_{m,l,n+1})) \quad (2.21)$$

$$L_1L_3U_{m,l,n} = -\frac{1}{8h^3Re}U_{m,l,n}\,\delta_x^3(U_{m,l,n} + U_{m,l,n+1}) \quad (2.22)$$

$$L_1L_4U_{m,l,n} = -\frac{1}{8hq^2Re}U_{m,l,n}\,\delta_x(\delta_y(\delta_y(U_{m,l,n} + U_{m,l,n+1}))) \quad (2.23)$$

$$L_2L_3U_{m,l,n} = -\frac{1}{8qh^2Re}V_{m,l,n}\,\delta_y(\delta_x^2(U_{m,l,n} + U_{m,l,n+1})) \quad (2.24)$$

$$L_2L_4V_{m,l,n} = -\frac{1}{8q^3Re}V_{m,l,n}\delta_y^3(U_{m,l,n} + U_{m,l,n+1}) \quad (2.25)$$

$$L_3L_4V_{m,l,n} = -\frac{1}{8q^3Re}V_{m,l,n}\delta_x^2\delta_y^2(U_{m,l,n} + U_{m,l,n+1}) \quad (2.26)$$

$$L_1{}^2\,U_{m,l,n} = \frac{1}{4h^2}U_{m,l,n}^2\delta_x^2(U_{m,l,n} + U_{m,l,n+1}) \quad (2.27)$$

$$L_2{}^2\,V_{m,l,n} = \frac{1}{4q^2}V_{m,l,n}^2\delta_y^2(U_{m,l,n} + U_{m,l,n+1}) \quad (2.28)$$

$$L_3{}^2\,U_{m,l,n} = \frac{1}{16Re^2h^4}\delta_x^4(U_{m,l,n} + U_{m,l,n+1}) \quad (2.29)$$

$$L_4{}^2\,U_{m,l,n} = \frac{1}{16Re^2q^4}\delta_y^4(U_{m,l,n} + U_{m,l,n+1}) \quad (2.30)$$

Using equations (2.17)-(2.30) in equation (2.16) and letting $q = h$, we obtain a discretization scheme by operator splitting.

2.3 Approximation at the Boundary

We use work developed by Kweyu (2012) for the initial and boundary conditions. We then use it on the derived numerical scheme to derive the solution.

The solution are given as:

$$u(x,y,t) = \frac{-2y-2\pi e^{-\frac{2\pi^2 t}{Re}}((\cos\pi x-\sin\pi x)\sin\pi y)}{Re(100+xy+e^{-\frac{2\pi^2 t}{Re}}((\cos\pi x-\sin\pi x)\sin\pi y)} \tag{2.31}$$

$$v(x,y,t) = \frac{-2x-2\pi e^{-\frac{2\pi^2 t}{Re}}((\cos\pi x-\sin\pi x)\sin\pi y)}{Re(100+xy+e^{-\frac{2\pi^2 t}{Re}}((\cos\pi x-\sin\pi x)\sin\pi y)} \tag{2.32}$$

and so

$$u_x = \frac{2\pi^2 e^{-\frac{2\pi^2 t}{Re}}((\sin\pi x+\cos\pi x)\sin\pi y)\left(Re\left(100+xy+e^{-\frac{2\pi^2 t}{Re}}((\cos\pi x-\sin\pi x)\sin\pi y)\right)\right) - (-2y-2\pi e^{-\frac{2\pi^2 t}{Re}}((\cos\pi x-\sin\pi x)\sin\pi y))(Re(y-\pi e^{-\frac{2\pi^2 t}{Re}}((\sin\pi x+\cos\pi x)\sin\pi y)))}{(Re(100+xy+e^{-\frac{2\pi^2 t}{Re}}((\cos\pi x-\sin\pi x)\sin\pi y))^2} \tag{2.33}$$

$$u_y = \frac{-2-2\pi^2 e^{-\frac{2\pi^2 t}{Re}}((\sin\pi x+\cos\pi x)\cos\pi y)\left(Re\left(100+xy+e^{-\frac{2\pi^2 t}{Re}}((\cos\pi x-\sin\pi x)\sin\pi y)\right)\right) - (-2y-2\pi e^{-\frac{2\pi^2 t}{Re}}((\cos\pi x-\sin\pi x)\sin\pi y))(Re(x-\pi e^{-\frac{2\pi^2 t}{Re}}((\sin\pi x-\cos\pi x)\cos\pi y)))}{(Re(100+xy+e^{-\frac{2\pi^2 t}{Re}}((\cos\pi x-\sin\pi x)\sin\pi y))^2} \tag{2.34}$$

$$v_x = \frac{-2-2\pi^2 e^{-\frac{2\pi^2 t}{Re}}((\sin\pi x+\cos\pi x)\cos\pi y)\left(Re\left(100+xy+e^{-\frac{2\pi^2 t}{Re}}((\cos\pi x-\sin\pi x)\sin\pi y)\right)\right) - (-2x-2\pi e^{-\frac{2\pi^2 t}{Re}}((\cos\pi x-\sin\pi x)\sin\pi y))(Re(y-\pi e^{-\frac{2\pi^2 t}{Re}}((\sin\pi x+\cos\pi x)\sin\pi y)))}{(Re(100+xy+e^{-\frac{2\pi^2 t}{Re}}((\cos\pi x-\sin\pi x)\sin\pi y))^2} \tag{2.35}$$

$$v_y = \frac{-2\pi^2 e^{-\frac{2\pi^2 t}{Re}}((\cos\pi x+\sin\pi x)\sin\pi y)\left(Re\left(100+xy+e^{-\frac{2\pi^2 t}{Re}}((\cos\pi x-\sin\pi x)\sin\pi y)\right)\right) - (-2x-2\pi e^{-\frac{2\pi^2 t}{Re}}((\cos\pi x-\sin\pi x)\sin\pi y))(Re(x-\pi e^{-\frac{2\pi^2 t}{Re}}((\sin\pi x-\cos\pi x)\cos\pi y)))}{(Re(100+xy+e^{-\frac{2\pi^2 t}{Re}}((\cos\pi x-\sin\pi x)\sin\pi y))^2} \tag{2.36}$$

Using forward finite difference to approximate equations (2.36) we have

$$\frac{U_{m+1,l,\omega}-U_{m,l,\omega}}{h} = \frac{2\pi^2 e^{-\frac{2\pi^2 t}{Re}}((\sin\pi x+\cos\pi x)\sin\pi y)\left(Re\left(100+xy+e^{-\frac{2\pi^2 t}{Re}}((\cos\pi x-\sin\pi x)\sin\pi y)\right)\right) - (-2y-2\pi e^{-\frac{2\pi^2 t}{Re}}((\cos\pi x-\sin\pi x)\sin\pi y))(Re(y-\pi e^{-\frac{2\pi^2 t}{Re}}((\sin\pi x+\cos\pi x)\sin\pi y)))}{(Re(100+xy+e^{-\frac{2\pi^2 t}{Re}}((\cos\pi x-\sin\pi x)\sin\pi y))^2} \tag{2.37}$$

$$\frac{U_{m,l+1,\omega}-U_{m,l,\omega}}{q} = \frac{-2-2\pi^2 e^{-\frac{2\pi^2 t}{Re}}((\sin\pi x+\cos\pi x)\cos\pi y)\left(Re\left(100+xy+e^{-\frac{2\pi^2 t}{Re}}((\cos\pi x-\sin\pi x)\sin\pi y)\right)\right) - (-2y-2\pi e^{-\frac{2\pi^2 t}{Re}}((\cos\pi x-\sin\pi x)\sin\pi y))(Re(x-\pi e^{-\frac{2\pi^2 t}{Re}}((\sin\pi x-\cos\pi x)\cos\pi y)))}{(Re(100+xy+e^{-\frac{2\pi^2 t}{Re}}((\cos\pi x-\sin\pi x)\sin\pi y))^2} \tag{2.38}$$

Using Newman's boundary conditions at the boundaries to approximate $U_{m\pm2,l,\omega}$ and $U_{m,l\pm2,\omega}$

At the $x = 0$ and the $y = 0$ boundaries, we have:

$$\frac{U_{m-1,l,\omega} - U_{m,l-2,\omega}}{h} = \pi e^{-\pi^2 \omega k}(0)$$

$$\frac{U_{m,l-1,\omega} - U_{m,l-2,\omega}}{q} = \pi e^{-\pi^2 \omega k}(0) \quad \text{respectively} \tag{2.39}$$

And so

$$U_{m-2,l,\omega} = U_{m-1,l,\omega}$$

$$U_{m,l-2,\omega} = U_{m,l-1,\omega} \tag{2.40}$$

At the $x = 1$ and the $y = 1$, boundaries

$$\frac{U_{m+2,l,\omega} - U_{m,l+1,\omega}}{h} = \pi e^{-\pi^2 \omega k}(0)$$

$$\frac{U_{m,l+2,\omega} - U_{m,l+1,\omega}}{q} = \pi e^{-\pi^2 \omega k}(0) \text{ respectively} \tag{2.41}$$

and so

$$U_{m+2,l,\omega} = U_{m+1,l,\omega}$$

$$U_{m,l+2,\omega} = U_{m,l+1,\omega} \tag{2.42}$$

In equations (2.37)-(2.42) $\omega = n$ or $n + 1$

For $\omega = n + 1$

$$U_{m-2,l,n+1} = U_{m-1,l,n+1}$$

$$U_{m,l-2,n+1} = U_{m,l-1,n+1} \tag{2.43}$$

and

$$U_{m+2,l,n+1} = U_{m+1,l,n+1}$$

$$U_{m,l+2,n+1} = U_{m,l+1,n+1} \tag{2.44}$$

For ω= n

$$U_{m-2,l,n} = U_{m-1,l,n}$$

$$U_{m,l-2,n} = U_{m,l-1,n} \tag{2.45}$$

and

$$U_{m+2,l,n} = U_{m+1,l,n}$$

$$U_{m,l+2,n} = U_{m,l+1,n} \tag{2.46}$$

Using equations (2.43)-(2.46) in equation developed in the previous section 2.1, we obtain the pure Crank-Nicholson scheme as shown below

$$\left(1 + 2kaU_{m,l,n} + 2kaV_{m,l,n} - 12kb - 4k^2a^2U_{m,l,n}V_{m,l,n} + 32k^2abU_{m,l,n} + 32k^2abV_{m,l,n} - 36k\right.$$
$$\left. 6ka^2U^2{}_{m,l,n} - 6kb^2V^2{}_{m,l,n} - 140b^2\right)U_{m,l,n+1} + \left(-kaU_{m,l,n} + 3kb + 2k^2a^2U_{m,l,n}V_{m,l,n} - \right.$$
$$\left. 16k^2abU_{m,l,n} - 6k^2abV_{m,l,n} + 18k^2b^2 + \frac{3}{2}ka^2U^2{}_{m,l,n} + \frac{35}{2}b^2\right)U_{m+1,l,n+1} + \left(-kaV_{m,l,n} + 3kb\right.$$
$$\left. 2k^2a^2U_{m,l,n}V_{m,l,n} - 16k^2abU_{m,l,n} - 6k^2abV_{m,l,n} + 18k^2b^2 + \frac{3}{2}kb^2V^2{}_{m,l,n} + \frac{35}{2}b^2\right)U_{m,l+1,n+1}$$
$$\left(-k^2a^2U_{m,l,n}V_{m,l,n} + 3k^2abU_{m,l,n} + 3k^2abV_{m,l,n} - 5k^2ab\right)U_{m+1,l+1,n+1} + \left(-kaU_{m,l,n} + 3kb + \right.$$
$$\left. 2k^2a^2U_{m,l,n}V_{m,l,n} - 16k^2abU_{m,l,n} - 6k^2abV_{m,l,n} + 18k^2ab + \frac{3}{2}ka^2U^2{}_{m,l,n} + \frac{35}{2}kb^2\right)U_{m-1,l,n+1}$$
$$\left(-kaV_{m,l,n} + 3kb + 2k^2a^2U_{m,l,n}V_{m,l,n} - 6k^2abU_{m,l,n} - 16k^2abV_{m,l,n} - 6k^2b^2 + \frac{3}{2}kb^2V^2{}_{m,l,n}\right.$$
$$\left. \frac{35}{2}b^2\right)U_{m,l-1,n+1} + \left(-k^2a^2U_{m,l,n}V_{m,l,n} + 3k^2abU_{m,l,n} + 3k^2abV_{m,l,n} - 9k^2ab\right)U_{m-1,l+1,n+1} +$$
$$\left(-k^2a^2U_{m,l,n}V_{m,l,n} + 3k^2abU_{m,l,n} + 3k^2abV_{m,l,n} - 13k^2ab\right)U_{m+1,l-1,n+1} + \left(-k^2a^2U_{m,l,n}V_{m,l,n}\right.$$
$$\left. 3k^2abU_{m,l,n} + 3k^2abV_{m,l,n} - 9k^2ab\right)U_{m-1,l-1,n+1} = -\left(2kaU_{m,l,n} + 2kaV_{m,l,n} - 12kb - \right.$$
$$4k^2a^2U_{m,l,n}V_{m,l,n} + 32k^2abU_{m,l,n} + 32k^2abV_{m,l,n} - 36k^2b^2 - 6ka^2U^2{}_{m,l,n} - 6kb^2V^2{}_{m,l,n} -$$
$$\left. 140b^2\right)U_{m,l,n} - \left(-kaU_{m,l,n} + 3kb + 2k^2a^2U_{m,l,n}V_{m,l,n} - 16k^2abU_{m,l,n} - 6k^2abV_{m,l,n} + 18k^2b\right.$$
$$\left. \frac{3}{2}ka^2U^2{}_{m,l,n} + \frac{35}{2}b^2\right)U_{m+1,l,n} - \left(-kaV_{m,l,n} + 3kb + 2k^2a^2U_{m,l,n}V_{m,l,n} - 16k^2abU_{m,l,n} - \right.$$
$$\left. 6k^2abV_{m,l,n} + 18k^2b^2 + \frac{3}{2}kb^2V^2{}_{m,l,n} + \frac{35}{2}b^2\right)U_{m,l+1,n} - \left(-k^2a^2U_{m,l,n}V_{m,l,n} + 3k^2abU_{m,l,n} + \right.$$
$$\left. 3k^2abV_{m,l,n} - 5k^2ab\right)U_{m+1,l+1,n} - \left(-kaU_{m,l,n} + 3kb + 2k^2a^2U_{m,l,n}V_{m,l,n} - 16k^2abU_{m,l,n} - \right.$$
$$\left. 6k^2abV_{m,l,n} + 18k^2ab + \frac{3}{2}ka^2U^2{}_{m,l,n} + \frac{35}{2}kb^2\right)U_{m-1,l,n} -$$
$$\left(-kaV_{m,l,n} + 3kb + 2k^2a^2U_{m,l,n}V_{m,l,n} - 6k^2abU_{m,l,n} - 16k^2abV_{m,l,n} - 6k^2b^2 + \frac{3}{2}kb^2V^2{}_{m,l,n}\right.$$
$$\left. \frac{35}{2}b^2\right)U_{m,l-1,n} - \left(-k^2a^2U_{m,l,n}V_{m,l,n} + 3k^2abU_{m,l,n} + 3k^2abV_{m,l,n} - 9k^2ab\right)U_{m-1,l+1,n} -$$
$$\left(-k^2a^2U_{m,l,n}V_{m,l,n} + 3k^2abU_{m,l,n} + 3k^2abV_{m,l,n} - 13k^2ab\right)U_{m+1,l-1,n} - \left(-k^2a^2U_{m,l,n}V_{m,l,n} + \right.$$
$$\left. 3k^2abU_{m,l,n} + 3k^2abV_{m,l,n} - 9k^2ab\right)U_{m-1,l-1,n} \qquad (2.47)$$

2.4 Crank-Nicholson-Lax-Friedrich's (CN-LF) Scheme.

For this scheme the first term $U_{m,l,n}$ in the right hand side of equation (2.47) is replaced by $\frac{1}{2}\left(U_{m+1,l+1,n} + U_{m-1,l+1,n}\right) + \frac{1}{2}\left(U_{m+1,l-1,n} + U_{m-1,l-1,n}\right)$.

3. RESULTS OF THE NUMERICAL SCHEMES DEVELOPED

We present the results using the following data: k=0.001, h=0.1, l=0.1. We now present the results. We shall display these results using tables and 3-D figures.

Table 1: Numerical Solution of u for Coupled Burgers' equation at $t = 1.0$, $y = 1.0$ and Re=5000.

x	Exact Solution u (*e-006)	Pure CN u (*e-006)	Hybrid CN-LF u (*e-006)
0.1	-0.3616507343010196	-0.3617871553843277	-0.3616370653493181
0.2	-0.7253219953726393	-0.7255883625253537	-0.7252953059518908
0.3	-1.090398562803828	-1.090790321153974	-1.090359309108974
0.4	-1.455935883797210	-1.45645150093615	-1.455884219315911
0.5	-1.820803748461822	-1.821445349043890	-1.820739460401386
0.6	-2.183867594804465	-2.184640978240838	-2.183790102318204
0.7	-2.544179042412541	-2.545093045825047	-2.544087460418207
0.8	-2.901144228647610	-2.902209524822090	-2.901037488242522
0.9	-3.254642313537360	-3.255869840866498	-3.254519319265866
1.0	-3.605076050695351	-3.606475329816048	-3.604935849144376

Table 2: Numerical Solution of v for Coupled Burgers' equation at $t = 1.0$, $y = 1.0$ and Re=5000.

x	Exact Solution v (*e-006)	Pure CN v (*e-006)	Hybrid CN-LF v (*e-006)
0.1	-3.972865925156529	-3.972759495591895	-3.972769188311192
0.2	-3.944368960551156	-3.944150135923467	-3.944170064848759
0.3	-3.913451475311227	-3.913110259949161	-3.913141335562608
0.4	-3.879306517566780	-3.878829975277790	-3.878873375785226
0.5	-3.841464954144779	-3.840838649848937	-3.840895689849297
0.6	-3.799845044283970	-3.799054658806107	-3.799126642198932
0.7	-3.754756190904309	-3.753789086686633	-3.753877164071588
0.8	-3.706856112188096	-3.705702631355121	-3.705807681840335
0.9	-3.657068620610165	-3.655722949394422	-3.655845501933214
1.0	-3.606475329816048	-3.604935849144376	-3.605076050695351

We provide a table of absolute errors and its line graph to give us a clear comparison. This is done in Table 3, Table 4, Graph 1 and Graph 2. The tables and graphs are self-explanatory.

Table 3: Absolute errors in Numerical Solution of u for Coupled Burgers' equation at $t = 1.0$, $y = 1.0$ and Re=5000.

x	Pure CN u (*e-006)	Hybrid CN-LF u (*e-006)
0.1	0.0001364210833079690	0.00001366895170101800
0.2	0.0002663671527139980	0.00002668942074901540
0.3	0.0003917583501500400	0.00003925369484991670
0.4	0.0005156171389399940	0.00005166448130000670
0.5	0.0006416005820699680	0.00006428806044000980
0.6	0.0007733834363699010	0.00007749248626032210
0.7	0.0009140034124999200	0.00009158199433967340
0.8	0.001065296174479700	0.0001067404050902890
0.9	0.001227527329129790	0.0001229942714999770
1	0.001399279120689820	0.0001402015509799350

Table 4: Absolute errors in Numerical Solution of v for Coupled Burgers' equation at $t = 1.0$, $y = 1.0$ and Re=5000.

x	CN (*e-006)	CN-LF (*e-006)
0.1	0.000106429564629806	0.000096736845329737
0.2	0.000218824627689962	0.000198895702399948
0.3	0.000341215362059888	0.000310139748620042
0.4	0.000476542288989634	0.000433141781559954
0.5	0.000626304295840097	0.000569264295480210
0.6	0.000790385477870359	0.000718402085040371
0.7	0.000967104217669768	0.000879026832719898
0.8	0.001153480832970290	0.001048430347760030
0.9	0.001345671215740030	0.001223118676950020
1	0.001539480671669760	0.001399279120689820

The above table shows that the CN-LF scheme provides accurate results closer to the exact solutions as compared to the CN scheme.

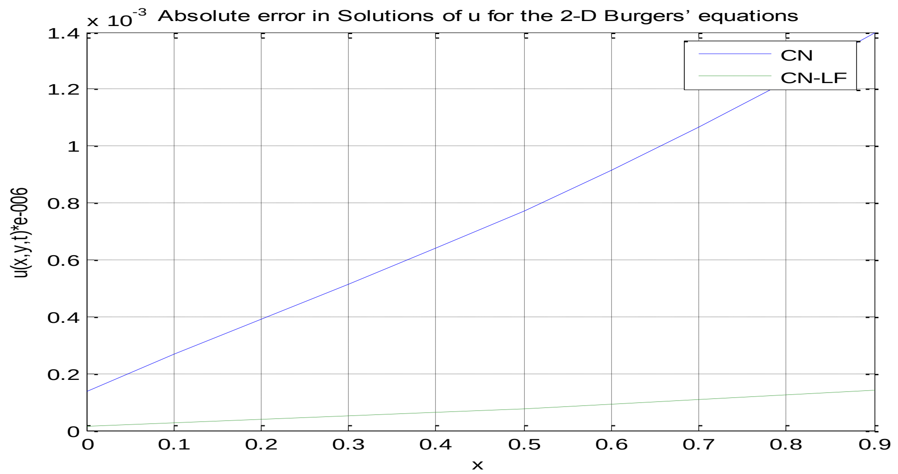

Figure 1: Absolute error in Solution of u for the 2-D Coupled Burgers' equation.

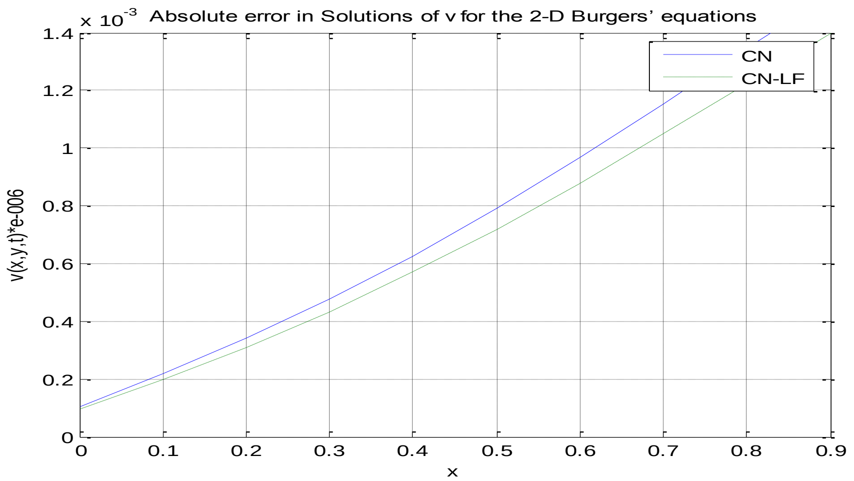

Figure 2: Absolute error in Solution of v for the 2-D Coupled Burgers' equation

Figure 1 and figure 2 clearly shows a decreased absolute error in CN-LF compared to CN for numerical solution of both u and v.

We now present 3-D solutions:

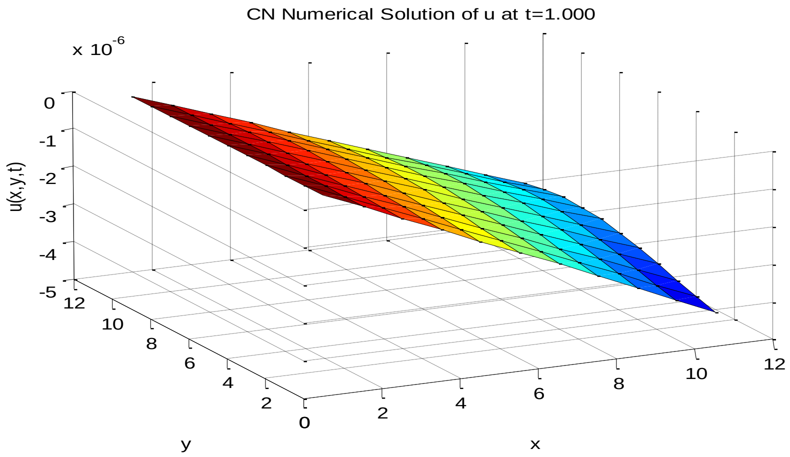

Figure 3: CN Numerical Solution of u at t=1.000

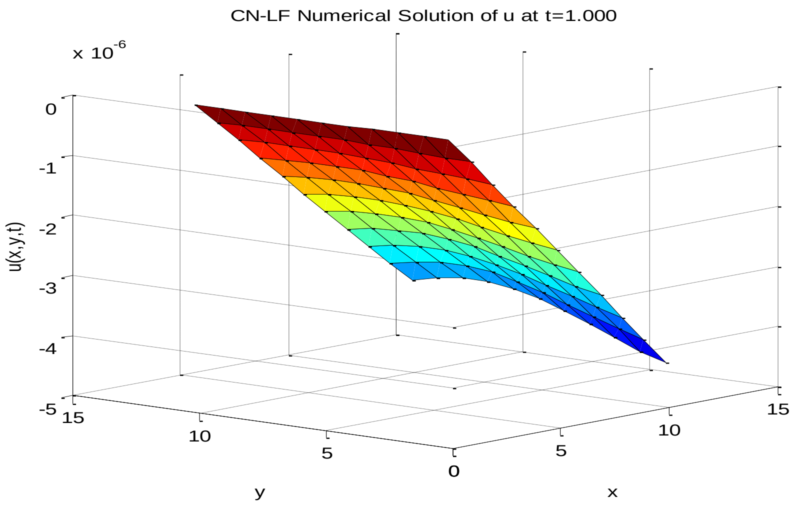

Figure 3: CN-LF Numerical Solution of u at t=1.000

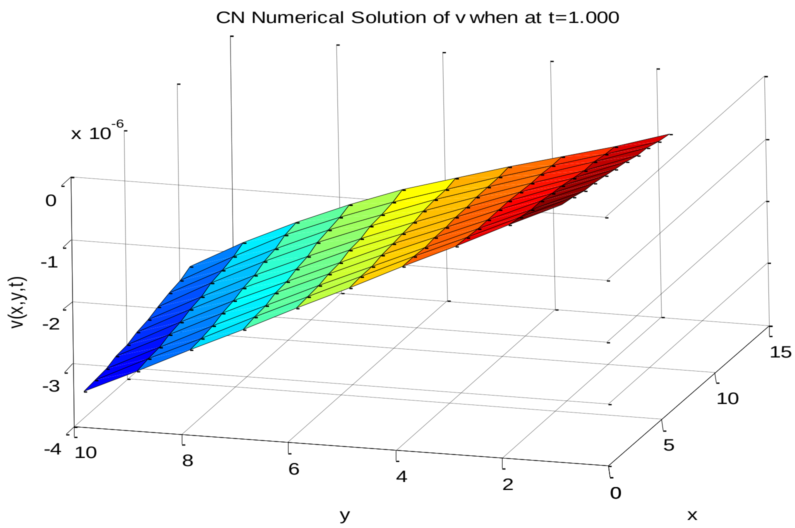

Figure 3: CN Numerical Solution of v at t=1.000

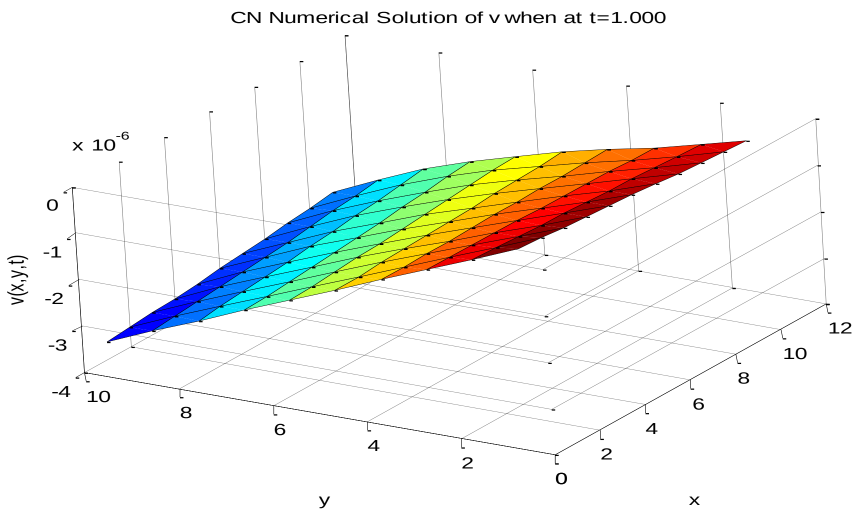

Figure 3: CN-LF Numerical Solution of v at t=1.000

We note that the 3-D solutions from all the methods developed take the same shape. It is thus established that the finite difference schemes developed are convergent.

4. CONCLUSION

The hybrid CN-LF scheme is the more accurate compared with the pure CN scheme with regard to the exact solution. The decrease in the absolute error also verifies the consistency of the scheme.

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