World Journal of Engineering Research and Technology
WJERT
www.wjert.org
SJIF Impact Factor: 3.419

ON THE STRUCTURES OF REGULAR SEMIRINGS

N. Sulochana*[1] and M. Amala[2]

[1]Dept of Mathematics, K.S.R.M College of Engineering, Kadapa, Andhra Pradesh, India.

[2]Dept. of Applied Mathematics, Yogi Vemana University, Kadapa, Andhra Pradesh, India.

Article Received on 06/08/2016 **Article Revised on 25/08/2016** **Article Accepted on 16/09/2016**

***Corresponding Author**
N. Sulochana
Dept of Mathematics, K.S.R.M College of Engineering, Kadapa, Andhra Pradesh, India.

ABSTRACT

In this paper we have established that If S is a boolean like semiring, PRD and (S, +) is p.t.o, then 1 is the maximum element.

AMS Mathematics Subject Classification (2010): 20M10, 16Y60.

KEYWORDS: Almost idempotent semiring, maximum, Periodic, Positive Rational Domain.

1. INTRODUCTION

There are several concepts of collective algebras to generalize that of a ring (R, +, •). Those are known as semirings, which begin from rings, generally speaking, by cancelling the assumption that (R, +) has to be a group. The semiring theory is attracting the concentration of a number of algebraists due to its applications to Formal language theory, Optimization theory, Computer Science, Automata theory, cryptography and the Mathematical Modeling of Quantum Physics. Especially semirings with different properties have become important in theoretical Computer Science.

The first formal definition of semiring was introduced in the year 1934 by Vandiver. However the developments of the theory in semirings have been taking place since 1950. Semirings flourish in the mathematical world around us. A semiring is basic structure in Mathematics.

The theory of ternary algebraic system was introduced by D.H.Lehmer. He investigated certain ternary algebraic systems called triplexes which turn out to be commutative ternary groups. The notion of ternary semigroups was introduced by S.Banach. He showed by an example that a ternary semigroup does not necessarily reduce to an ordinary semigroup. In W.G.Lister characterized additive subgroups of rings which are closed under the triple ring product and he called this algebraic system a ternary ring. T.K.Dutta and S.Kar introduced and studied some properties of ternary semirings which is a generalization of ternary ring.

Ternary semiring arises naturally as follows; consider the ring of integers Z which plays a vital role in the theory of ring. The subset $Z+$ of all positive integers of Z is an additive semigroup which is closed under the ring product, i.e. $Z+$ is a semiring. Now, if we consider the subset $Z-$ of all negative integers of Z, then we see that $Z-$ is an additive semigroup which is closed under the triple ring product (however, $Z-$ is not closed under the binary ring product), i.e. $Z-$ forms a ternary semiring. Thus we see that in the ring of integers Z, $Z+$ forms a semiring whereas $Z-$ forms a ternary semiring. $Z+$ forms a semiring where as $Z-$ forms a ternary semiring in the fuzzy settings too.

Section one deals with completely regular ternary semiring. Section two contains some special classes of semirings with different constrains.

2. Preliminaries

Definition 2.1

An element a of S is completely regular ternary if there exists x in S satisfying the following conditions.

(A) $a = a + x + a$ and

(B) $[aa(a + x)] = a + x$.

Thus S is a completely regular ternary semiring if every element of S is completely regular ternary.

Definition 2.2

A semiring S is called b-lattice if (S, +) is semilattice and (S, ·) is a band.

Definition 2.3

A semiring S is called a commutative ternary semiring if [abc] = [bac] = [bca] for all a, b, c in S.

Definition 2.4

An element a in completely regular ternary semiring S is called idempotent if [aaa] = a.

Definition 2.5

An additive commutative semigroup S, together with a ternary multiplication denoted by [ ] is said to be a ternary semiring if

i) [[abc] de] = [a [bcd] e] = [ab [cde]]

ii) [(a + b) cd] = [acd] + [bcd]

iii) [a (b + c)d] = [abd] + [acd] and

iv) [ab (c + d)] = [abc] + [abd] for all a, b, c, d, e in S.

Definition 2.6

A semigroup (S, +) is semilattice if (S, +) is band and commutative.

i.e a + a = a and a + x = x + a.

Definition 2.7

A semigroup (S, •) is left (right) singular if ab = a (ab = b) for all a, b in S.

A semigroup (S, +) is left (right) singular if a + b = a (a + b = b) for all a, b in S.

Definition 2.8

A semiring S is said to be boolean like semiring, if (i) ab (a + b + ab) = ab for all a, b in S and a.0 = 0.a = 0 (ii) Weak commutative.

Definition 2.9

A viterbi semiring is a semiring in which S is additively idempotent and multiplicatively subidempotent.

i.e., a + a = a and a + a^2 = a for all 'a' in S.

Definition 2.10

A semiring S is almost idempotent if a + a^2 = a^2 for all a in S.

Definition 2.11

An element a in a semigroup (S, +) is periodic if ma = na where m and n are positive integers. A semigroup (S, +) is periodic if every one of its elements is periodic.

Definition 2.12

A semiring S is Positive Rational Domain (PRD) if (S, •) is an abelian group.

Definition 2.13

The zeroid of a semiring S is $\{x \in S \mid z + x = z \text{ or } x + z = z \text{ for some } z \in R\}$.

Definition 2.14

In a totally ordered semiring $(S, +, \bullet, \leq)$, $(S, +, \leq)$ is positively totally ordered (p.t.o), if $a + b \geq a, b$ for all a, b in S.

Definition 2.15

A semiring S is zero square if $a^2 = 0$ for all a in S.

3. Structures of Completely Regular Ternary Semiring.

Proposition 3.1: Let S be a b-lattice semiring. Then S is completely regular ternary semiring if and only if $a + x = a$.

Proof: Assume that S is a b-lattice semiring implies that $[aaa] = a$

Suppose $a + x = a$ which implies $a + x + a = a + a$ then $a + x + a = a$ ----- (I)

By hypothesis $a + x = a$

Consider $[aa(a + x)] = [aaa] = a$

$\Rightarrow [aa(a + x)] = a + x$ ------ (II)

From (I) and (II), S is completely regular ternary semiring

Conversely, let us suppose $a + x + a = a$ that implies $a + a + x = a$

Thus $a + x = a$

Theorem 3.2: Suppose S is a completely regular ternary semiring. Then $(S, \cdot)$ is idempotent in the following cases

(i) $(S, +)$ is commutative and 'e' is the multiplicative which is also additive identity.

(ii) $(S, +)$ is semilattice.

(iii)$(S, +)$ is right singular and $(S, +)$ is right cancellative.

Proof: (i) From the definition of completely regular ternary semiring, $a + x + a = a$

By using $(S, +)$ commutative we get $a + a + x = a$

$\Rightarrow a.e + a.e + x = a$

This leads to $a + x = a$

From condition (B), $[aa(a + x)] = a + x$ becomes as $[aaa] = a$

Therefore $(S, \cdot)$ is idempotent

(ii) Assume that $(S, +)$ is semilattice

Let us take $a + x + a = a$ which can be written as $a + a + x = a$

The above equation becomes $a + x = a$

Similar procedure is carried out as above

(iii) By hypothesis $(S, +)$ is right singular, $a + x = x$ --- (1)

Also from condition (B) $[aa (a + x)] = a + x$

Using first equation in above we get $[aax] = x$

Again $[aa (a + x)] = a + x$

The above equation also written as $[aaa] + [aax] = a + x$

$\Rightarrow [aaa] + x = a + x$

Since $(S, +)$ is right cancellative this implies $[aaa] = a$

Therefore $(S, \cdot)$ is idempotent

Proposition 3.3: Let S be a boolean like semiring and $(S, \cdot)$ be left singular semigroup then $(S, +)$ is periodic.

Proof: Consider $ab (a + b + ab) = ab$

By hypothesis $(S, \cdot)$ is left singular implies $ab = a$ for all a, b in S

Therefore $a (a + b + a) = a$ it becomes as $a^2 + ab + a^2 = a$

Since $(S, \cdot)$ is left singular $a + a + a = a$

$\Rightarrow 3a = a$

Hence $(S, +)$ is periodic

Theorem 3.4: If S is a boolean like semiring then $(S, \cdot)$ is periodic in the following cases.

(i) viterbi semiring

(ii) Almost idempotent semiring.

Proof: **(i)** First we consider boolean like semiring $aa (a + a + aa) = aa$ for all a in S

From the definition of viterbi semiring $a + a = a$ and $a + a^2 = a$

Then the above equation takes the form $a^2 (a + a^2) = a^2$

This implies $a^3 = a^2$

Therefore $(S, \cdot)$ is periodic

(ii) Suppose S is almost idempotent semiring.

We consider boolean like semiring aa (a + a + aa) = aa for all a in S

$\Rightarrow a^2 (a + a^2) = a^2$

$\Rightarrow a^4 = a^2$

Thus (S, ·) is periodic

Theorem 3.5: If S is a Positive Rational Domain.

(i) If Z is a zeroid of S then S = Z.

(ii) If S is a boolean like semiring and (S, +) is p.t.o then 1 is the maximum element.

Proof: (i) Let $a \in Z$ where Z is a zeroid of S. then there exists $b \in S$ such that a + b = b or b + a = b

On multiplication of b we have $(a + b)\, b = b^2$

this implies $ab + b^2 = b^2$

$\Rightarrow ab \in Z$

Suppose $(b + a)b = b^2 \Rightarrow b^2 + ab = b^2 \Rightarrow ab \in Z$

Therefore ab is a zeroid

Since S is PRD then (S, ·) is commutative thus 'ba' is a zeroid

Thus Z is an ideal of S

Suppose $a + b = b \Rightarrow a^{-1}(a + b) = a^{-1}b$

$\Rightarrow 1 + a^{1}b = a^{-1}b$

i.e., 1 + s = s, where $s = a^{-1}b$

$\Rightarrow 1 \in Z$

Which implies 1 is a zeroid

If $a \in S$ then $a = a.1 \in Z$ implies $S \subseteq Z$ But $Z \subseteq S$

Therefore S = Z

(ii) Consider ab (a + b + ab) = ab

Since S is PRD, a + b + ab = 1

This implies $1 = a + b + ab \geq a$, $1 = a + b + ab \geq b$ and $1 = a + b + ab \geq ab$

Which implies $1 \geq a$, $1 \geq b$ and $1 \geq ab$

Therefore 1 is the maximum element

4. Some Special Structures of Semirings

Theorem 4.1: Let S be a semiring containing additive identity which is also multiplicative identity and if (S, +) is right cancellative. Then (S, ·) is a band.

Proof: Assume that S is a semiring which contains additive identity which is also multiplicative identity.

Let a, 1 are elements of S then 1 + 1 = 1 and a + 1 = a

$\Rightarrow$ a + a = a and $a + a^2 = a^2$

$\Rightarrow a^2 + a^2 = a^2$ and $a + a^2 = a^2$

$\Rightarrow a^2 + a^2 = a + a^2$

Using (S, +) is right cancellation

This implies $a^2 = a$ for all a in S

Hence (S, ·) is band

Theorem 4.2: Let S be a zero square semiring where 0 is the additive identity in which $ab + a^1 b^1 = a b^1 + a^1 b$ for all a, b, a^1, b^1 in S. Then

(i) $aa^1bb^1 = a^1ab^1b = 0$

(ii) $a b = a b^1$ and $a^1 b^1 = a^1 b$. If (S, ·) is left cancellative

Proof: **(i)** First let us consider $ab + a^1 b^1 = a b^1 + a^1 b$ for all a, b in S

$\Rightarrow a (ab + a^1 b^1) = a (a b^1 + a^1 b)$

$\Rightarrow a^2 b + a a^1 b^1 = a^2 b^1 + a a^1 b$

By the definition of zero square semiring, $a^2 = 0$ for all a in S

Then above equation becomes 0 + a.a.1 b.1 = 0 + a.a.1.b

$\Rightarrow a a^1 b^1 = a a^1 b$

$\Rightarrow a a^1 b^1 b^1 = a a^1 bb^1$

$\Rightarrow a a^1 b^1 b = 0$

Similarly let us take $ab + a^1 b^1 = a b^1 + a^1 b$ for all a, b in S

$\Rightarrow (ab + a^1 b^1) b = (a b^1 + a^1 b) b$

$\Rightarrow a b^2 + a^1 b^1 b = a b^1 b + a^1 b^2$

Since S is a zero square semiring $b^2 = 0$ for all b in S

$\Rightarrow 0 + a^1 b^1 b = a b^1 b + 0$

$\Rightarrow a^1 b^1 b = a b^1 b$

$\Rightarrow$ $a^1 a^1 b^1 b = a^1 a b^1 b$

$\Rightarrow$ $a^1 a b^1 b = 0$

(ii) Consider $ab + a^1 b^1 = a b^1 + a^1 b$ for all a, b in S

$\Rightarrow a^1 (ab + a^1 b^1) = a^1 (a b^1 + a^1 b)$

$\Rightarrow a^1 a b + (a^1)^2 b^1 = a^1 a b^1 + (a^1)^2 b$

$\Rightarrow a^1 a b + 0 = a^1 a b^1 + 0$

$\Rightarrow a^1 a b = a^1 a b^1$

Using (S,) left cancellation law in above it becomes $ab = a b^1 \rightarrow (1)$

Similarly $ab + a^1 b^1 = a b^1 + a^1 b$ for all a, b in S

$\Rightarrow a (ab + a^1 b^1) = a (a b^1 + a^1 b)$

$\Rightarrow a^2 b + a a^1 b^1 = a^2 b^1 + a a^1 b$

$\Rightarrow 0 + a a^1 b^1 = 0 + a a^1 b$

$\Rightarrow a a^1 b^1 = a a^1 b$

$\Rightarrow a^1 b^1 = a^1 b \rightarrow (2)$

From (1) and (2) $a b = a b^1$ and $a^1 b^1 = a^1 b$

REFERENCES

1. V.R.Daddi and Y.S.Pawar, "On Completely Regular Ternary Semiring", Novi Sad J. Math., 2012; 42(2): 1-7.
2. Jonathan S.Golan, "Semirings and their Applications", Kluwer Academic Publishers., 1999.
3. Jonathan S.Golan, "Semirings and Affine Equations over Them: Theory and Applications". Kluwer Academic Publishers., 1999.
4. K.Hailu, B.Belayneh and K.Venkateswarlu, "Boolean Like Semi Ring of Fractions" Teshome International Journal of Mathematical Archive., April – 2012; 3(4): 1554–1560.
5. B.V.N.Murthy and K.Venkateswarlu, "Contraction of Ideals In Boolean Like Semiring", International Journal of Mathematical Archive., 2012; 3(4): 1324-1328.
6. M.Satyanarayana, "Positively ordered semigroups", Lecture notes in Pure and Applied Mathematics, Marcel Dekker, Inc., 1979; 42.
7. M.Satyanarayana, "On the additive semigroup structure of semirings", Semigroup Forum, 1981; 23: 7-14.

8. M. Satyanarayana, “On the additive semigroup of ordered semirings”, Semigroup Forum, 1985; 31: 193-199.
9. M.K.Sen, S.K.Maity and K.P.Shum, “Clifford Semirings and Generalized Clifford Semirings” Taiwanese Journal of Mathematics, September 2005; 9(3): 433-444.
10. K. Venkateswarlu, B.V.N.Murthy and N.Amarnath “Boolean Like Semirings Int.J. Contemp. Math Sciences, 2011; 6(13): 619–635.