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# ON THE BINARY QUADRATIC EQUATION $ax^2 - (a+1)y^2 = a$

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**ABSTRACT**

The binary quadratic equation $ax^2 - (a+1)y^2 = a$ represents a hyperbola. In this paper we obtain a sequence of its integral solutions and present a few interesting relations among them.

**KEYWORDS:** Binary quadratic, Hyperbola, Parabola, Integral solutions, Pell equation. 2010 Mathematics subject classification: 11D09.

## INTRODUCTION

The binary quadratic Diophantine equations (both homogeneous and non-homogeneous) are rich in variety.[1-5] In[6-12] the binary quadratic non-homogeneous equations representing hyperbolas respectively are studied for their non-zero integral solutions. These results have motivated us to search for infinitely many non-zero integral solutions of another interesting binary quadratic equation given by $ax^2 - (a+1)y^2 = a$. The recurrence relations satisfied by the solutions $x$ and $y$ are given. Also a few interesting properties among the solutions are exhibited.

## METHOD OF ANALYSIS

The Diophantine equation representing the binary quadratic equation to be solved for its non-zero distinct integral solution is

$$ax^2 - (a+1)y^2 = a \tag{1}$$

Substituting the linear transformations

$$x = X \pm (a+1)T \quad , \quad y = X \pm aT \tag{2}$$

in (1) , we have

$$X^2 = a(a+1)T^2 - a \tag{3}$$

The least positive integer solution is

$$T_0 = 1, X_0 = a \tag{4}$$

Now , to find the other solutions of (3) , consider the pellian equation

$$X^2 = a(a+1)T^2 + 1 \tag{5}$$

whose fundamental solution is

$$\left(\tilde{T}_0, \tilde{X}_0\right) = (2, 2a+1)$$

The other solutions of (6) can be derived from the relations

$$\tilde{X}_n = \frac{f_n}{2}$$

$$\tilde{T}_n = \frac{g_n}{2\sqrt{a^2+a}}$$

where

$$f_n = \left(2a+1+2\sqrt{a^2+a}\right)^{n+1} + \left(2a+1-2\sqrt{a^2+a}\right)^{n+1}$$
$$g_n = \left(2a+1+2\sqrt{a^2+a}\right)^{n+1} - \left(2a+1-2\sqrt{a^2+a}\right)^{n+1}$$

Applying the Brahmagupta lemma between $(T_0, X_0)$ and $\left(\tilde{T}_n, \tilde{X}_n\right)$, the other solutions of (3) can be obtained from the relation

$$\begin{aligned} T_{n+1} &= \frac{1}{2} f_n + \frac{a}{2\sqrt{a^2+a}} g_n \\ X_{n+1} &= \frac{a}{2} f_n + \frac{\sqrt{a^2+a}}{2} g_n \end{aligned} \tag{6}$$

By substituting equation (6) in (2), the non-zero distinct integer solutions of (1) are obtained as follows

$$x_{n+1} = \frac{-1}{2} f_n \ , \left[\frac{2a+1}{2} f_n + \sqrt{a^2+a}\, g_n\right]$$

$$y_{n+1} = \frac{\sqrt{a^2+a}}{2(a+1)} g_n \ , \left[a\, f_n + \frac{(2a+1)\sqrt{a^2+a}}{2(a+1)} g_n\right]$$

The recurrence relations for $x_{n+1}, y_{n+1}$ are respectively

$$x_{n+3} - (4a+2)x_{n+2} + x_{n+1} = 0$$
$$y_{n+3} - (4a+2)y_{n+2} + y_{n+1} = 0.$$

From the above solutions we obtain some interesting relations, which are presented below:

1. Relations among the solutions:

- $2x_{n+3} = (8a+4)x_{n+2} - 2x_{n+1}$
- $2(2a+1)x_{n+2} = x_{n+1} + 2x_{n+3}$
- $x_{n+2} = x_{n+1} - 2(a+1)y_{n+2}$
- $2(a+1)y_{n+2} = x_{n+1} - (2a+1)x_{n+2}$
- $(8a^2+8a+1)x_{n+2} = (2a+1)x_{n+1} - 2(a+1)y_{n+3}$
- $(2a+1)y_{n+1} = y_{n+2} + 2ax_{n+1}$
- $4(a+1)(2a+1)y_{n+1} = (8a^2+8a+1)x_{n+1} - 2x_{n+3}$
- $(8a^2+8a+1)y_{n+2} = 2ax_{n+1} + (2a+1)y_{n+3}$
- $4(a+1)y_{n+2} = x_{n+1} - 2x_{n+3}$
- $2(a+1)y_{n+3} = (2a+1)x_{n+1} - (8a^2+8a+1)x_{n+2}$
- $2(a+1)y_{n+1} = (2a+1)x_{n+1} - x_{n+2}$
- $x_{n+3} = x_{n+1} - 4(a+1)y_{n+2}$
- $4(a+1)(2a+1)y_{n+3} = x_{n+1} - 2(8a^2+8a+1)x_{n+3}$
- $(8a^2+8a+1)y_{n+1} = 4a(2a+1)x_{n+1} + y_{n+3}$
- $(2a+1)y_{n+3} = (8a^2+8a+1)y_{n+2} - 2ax_{n+1}$
- $(8a^2+8a+1)x_{n+3} = x_{n+1} - 4(a+1)(2a+1)y_{n+3}$
- $2ax_{n+1} = (2a+1)y_{n+1} - y_{n+2}$
- $x_{n+1} = 2(2a+1)x_{n+2} - x_{n+3}$
- $2ax_{n+2} = y_{n+1} - (2a+1)y_{n+2}$
- $2(4a^2+5a+1)(8a^2+8a+1)y_{n+1} = (8a^2+8a+1)x_{n+2} - (2a+1)x_{n+3}$
- $2y_{n+3} = 4(2a+1)y_{n+2} - 2y_{n+1}$
- $2ax_{n+3} = (2a+1)y_{n+1} - (8a^2+8a+1)y_{n+2}$
- $$\left(8a^2+8a+1\right)\left\{(8a^2+10a+2)y_{n+2} + (64a^4+112a^3+56a^2+6a-1)x_{n+2}\right\} = \left(128a^5+288a^4+208a^3+48a^2-1\right)x_{n+3}$$

**OBSERVATIONS**

**I.** Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of hyperbolas which are presented below.

**Hyperbolas**

- $4(a^2+a)[x_{n+1}^2-1]-[(2a+1)x_{n+1}-x_{n+2}]^2=0.$
- $16(2a+1)^2(a^2+a)[x_{n+1}^2-1]-[(8a^2+8a+1)x_{n+1}-2x_{n+3}]^2=0.$
- $4(2a+1)^2(a^2+a)[x_{n+1}^2-1]-(a+1)^2[2y_{n+2}+4ax_{n+1}]^2=0.$
- $4(8a^2+8a+1)^2(a^2+a)[x_{n+1}^2-1]-(a+1)^2[8a(2a+1)x_{n+1}+2y_{n+3}]^2=0.$
- $(4a+1)^2(a^2+a)(8a^2+8a+1)^2\{[2x_{n+3}-4(2a+1)x_{n+2}]^2-2^2\}-[(8a^2+8a+1)x_{n+2}-(2a+1)x_{n+3}]^2=0.$
- $[y_{n+2}-(2a+1)y_{n+1}]^2-4a(a+1)y_{n+1}^2=4a^2.$
- $[y_{n+3}-(8a^2+8a+1)y_{n+1}]^2-16a(2a+1)[(a+1)y_{n+1}^2+a]=0.$

**II.** Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of parabolas which are presented below.

***Parabolas***

- $2a(a+1)[x_{2n+2}+1]+[(2a+1)x_{n+1}-x_{n+2}]=0.$
- $[y_{2n+3}-(2a+1)y_{2n+2}+2a]-4(a+1)y_{n+1}^2=4a.$
- $8a(a+1)(2a+1)^2[1+x_{2n+2}]+[(8a^2+8a+1)x_{n+1}-2x_{n+3}]^2=0.$
- $2a(a+1)(4a+1)^2(8a^2+8a+1)^2[x_{2n+4}-2(2a+1)x_{2n+3}-1]-[(8a^2+8a+1)x_{n+2}-(2a+1)x_{n+3}]^2=0.$

  $2a(a+1)(8a^2+8a+1)^2[1+x_{2n+2}]+(a+1)^2[8a(2a+1)x_{n+1}+2y_{n+3}]^2=0.$
- $2a(a+1)(2a+1)^2[1+x_{2n+2}]+(a+1)^2[2y_{n+2}+4ax_{n+1}]^2=0.$

**Generation of Solution**

If $x_1=x_0+h$ and $y_1=h-y_0$ is any solution of (1) and we have the following $x_0, y_0$ also satisfies (1).

Let $x_1=x_0+h$ , $y_1=h-y_0$ and $h\neq 0$ (7)

Substituting (7) in (1) and performing a few calculations , we obtain

$$h=2ax_0+2(a+1)y_0$$

and then

$$x_1 = (2a+1)x_0 + (2a+2)y_0$$
$$y_1 = 2ax_0 + (2a+1)y_0$$

which is written in the form of matrix as

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = M \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

where $M = \begin{pmatrix} 2a+1 & 2a+2 \\ 2a & 2a+1 \end{pmatrix}$

replacing the above process, the general solution $(x_n, y_n)$ to (1) is given by

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = M^n \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

The eigen values of Mare $\alpha = (2a+1) + 2\sqrt{a^2+a}$ and $\beta = (2a+1) - 2\sqrt{a^2+a}$ , it is well known that

$$M^n = \frac{\alpha^n}{\alpha-\beta}(M-\beta I) + \frac{\beta^n}{\alpha-\beta}(M-\alpha I)$$

Using the above formula, we have

$$M^n = \begin{pmatrix} \dfrac{\alpha^n+\beta^n}{2} & \dfrac{(a+1)}{2\sqrt{a^2+a}}(\alpha^n-\beta^n) \\ \dfrac{a}{2\sqrt{a^2+a}}(\alpha^n-\beta^n) & \dfrac{\alpha^n+\beta^n}{2} \end{pmatrix}$$

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} Y_n & (a+1)X_n \\ aX_n & Y_n \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

where

$$Y_n = \frac{1}{2} f_n \qquad , f_n = \alpha^n + \beta^n$$

$$X_n = \frac{1}{2\sqrt{a^2+a}} g_n, g_n = \alpha^n - \beta^n$$

**Remarkable observations**

Let $(\alpha, \beta, \gamma)$ be the sides of the Pythagorean triangle

$$\alpha = 2pq, \beta = p^2 - q^2, \gamma = p^2 + q^2, p > q > 0$$

where p and q are the generators of the Pythagorean triangle.

Let A and P be its area and perimeter respectively.

Write p and q as $p = x_n + y_n$ and $q = y_n$, where $(x_n, y_n)$ is the solution of (1).

Then the corresponding Pythagorean triangle is such that

- $\gamma(a-1) - 2a\alpha + (a+1)\beta = 2a.$
- $aP^2 + P[(a+1)(\beta - \gamma) - 2a(\alpha+1)] = 4aA.$

## CONCLUSION

To conclude, one may search for other patterns of solutions and their corresponding properties.

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