



JUST TOTAL EXCELLENT DOMINATION IN FUZZY GRAPHS

¹*P. Nithya and ²K. M. Dharmalingam

²Department of Mathematics, The Madura College, Madurai.

Article Received on 10/09/2017

Article Revised on 01/10/2017

Article Accepted on 22/10/2017

*Corresponding Author

P. Nithya

Department of Mathematics,
The Madura College,
Madurai.

ABSTRACT

Let G be a fuzzy graph. A subset D of V is said to be fuzzy dominating set if every vertex $u \in V(G)$ there exist there exist a vertex $v \in V - D$ such that $uv \in E(G)$ and $\mu(uv) \leq \sigma(u) \wedge \sigma(v)$. The minimum cardinality of fuzzy dominating set is denoted by $\gamma^f(G)$. A

Fuzzy graph G is said to be Fuzzy excellent if for every vertex of G belongs to γ^f -set of G . In this paper, we introduced a new class of total fuzzy excellent and just total fuzzy excellent (JTFE). Also, in this paper we initiate to study an induced subgraph of a total fuzzy excellent graph and we obtain a necessary and sufficient condition for a graph to be just total fuzzy excellent. We find an upper bound for $\gamma_i^f(G)$. We also prove that every just total fuzzy excellent graph contains no cut vertex.

KEYWORD: Total fuzzy dominating set, Total fuzzy excellent, Just total fuzzy excellent

Subject Classification: 05C72.

1. INTRODUCTION

Fuzzy graphs were introduced by Rosenfeld.^[9] Rosenfeld has described the fuzzy analogue of several graph theoretic concept like paths, cycle, tree, and connectedness and established some properties on them. A.Somasundaram and S.Somasundaram introduced total domination in fuzzy graphs using effective edges.^[3] It is further studied by Depnath.^[10] Prof.N.Sridharan and M.Yamuna have introduced the concepts of just excellence and very excellence graph.^[2] The notation of domination in fuzzy graphs has been growing very fast and has numerous application in various fields. Here we introduced the concept of just total excellent domination in fuzzy graphs.

2. PRELIMINARIES

Definition: 2.1

A fuzzy graph $G = (\sigma, \mu)$ is a pair of function

$\sigma: V \rightarrow [0,1]$ & $\mu: VXV \rightarrow [0,1]$ where for all $u, v \in V$ we have $\mu(u, v) \leq \sigma(u) \wedge \sigma(v)$

Definition: 2.2

The order p and size q of the fuzzy graph $G = (\sigma, \mu)$ are defined to be

$$p = \sum_{x \in V} \sigma(x) \text{ \& } q = \sum_{xy \in E} \mu(xy)$$

Definition: 2.3

The domatic number $d^f(G)$ of graph G is defined to be the maximum number of elements in a partition of $V(G)$ into dominating sets.

Definition: 2.4

The subset D of V is said to be fuzzy dominating set if every vertex $u \in V(G)$ there exist a vertex $v \in V - D$ such that $uv \in E(G)$ and $\mu(uv) \leq \sigma(u) \wedge \sigma(v)$. The minimum cardinality of fuzzy dominating set is denoted by $\gamma^f(G)$.

Definition: 2.5

Let G be a fuzzy graph. A subset S of G is said to be fuzzy independent set of G if there exists no $uv \in S$ such that $\mu(uv) \leq \sigma(u) \wedge \sigma(v)$. The maximum cardinality of such fuzzy independent set is called fuzzy independence number and is denoted by β_0^f .

Definition: 2.6

A Clique of a simple graph G is subset S of V such that $G[S]$ is complete. The number of vertices in a largest clique of G is called the clique number of G and is denoted by $w(G)$.

Definition: 2.7

The private neighborhood in fuzzy graph G is denoted by $PN^f(v, S)$ and is defined as $N^f(V) - N^f[S - \{v\}]$ where $uv \in E(G)$ such that $\mu(uv) \leq \sigma(u) \wedge \sigma(v)$ and each $u \in PN^f(v, S)$.

Definition: 2.8

A fuzzy graph G is said to be fuzzy excellent if for every vertex of G belongs to γ^f -set of G .

Definition: 2.9

A fuzzy graph G is said to be just fuzzy excellent graph if every vertex of G appears in a unique γ^f -sets of G .

3. MAIN DEFINITIONS AND RESULTS**Definition 3.1: Total fuzzy dominating set**

Let S be the fuzzy minimal dominating set of G . Then the set S is a total fuzzy dominating set if $N[S] = V$ (i.e) $N[S] = \{u \in S \mid uv \in E, u \neq v \ \& \ \mu(uv) \leq \sigma(u) \wedge \sigma(v) \forall v \in V\} = V$

Definition 3.2: Total fuzzy excellent

A graph G is said to be total fuzzy excellent graph if to each $u \in V$ there is a total fuzzy dominating set γ_t^f -set of G containing u .

Theorem: 3.3

Let G be the connected fuzzy graph G , then there exist a total fuzzy excellent graph H contains G as an induced subgraph.

Proof

Let u_0, u_1, \dots, u_{n-1} be the vertices of G and $u_i u_j \in E(G)$ such that $\mu(u_i u_j) \leq \sigma(u_i) \sigma(u_j)$.

Consider the cycle C_{4n} with vertices v_1, v_2, \dots, v_{4n} .

Let $E(G) = \{v_{4i+1} v_{4j+1} \mid u_i u_j \in E(G) \text{ and } \mu(v_{4i+1} v_{4j+1}) \leq \sigma(v_{4i+1}) \wedge \sigma(v_{4j+1})\}$. Construct a graph H with $V(H) = \{v_i \mid 1 \leq i \leq 4n\}$ and $E(H) = E(C_{4n}) \cup E'$. Thus the resulting graph H is total fuzzy excellent containing as an included subgraph.

Definition 3.4: Just total fuzzy excellent (JTFE)

A graph G is said to be just total fuzzy excellent if to each $u \in V$ there exist a unique γ_t^f -set of G containing u .

Remark 3.5

1. Every JTFE is total fuzzy excellent
2. If G is JTFE, then $\delta(G) \geq n/\gamma_t^f(G)$.

Proof

Let $V = S_1 \cup S_2 \cup \dots \cup S_m$ be the partition of V into γ_t^f -sets of G . Fix one $u \in V$. Assume that $u \in S_j$. Since each S_i is a γ_t^f -set, u is adjacent to atleast one vertex of S_i . Hence $\delta(u) \geq m = n/\gamma_t^f(G)$.

Theorem 3.6

A graph G is JTFE if and only if

- i) $\gamma_t^f(G)$ divides n
- ii) $d_t^f(G) = \frac{n}{\gamma_t^f(G)}$
- iii) G has exactly $\frac{n}{\gamma_t^f(G)}$ distinct γ_t^f -sets.

Definition 3.7

If D is total fuzzy dominating set of G , for each $u \in D$, the total fuzzy private neighbor of u is defined as $PN_t^f(u, D) = \{v \in V / N(v) \cap D = \{u\}\}$.

If D is a minimal total fuzzy dominating set of G , then $PN_t^f(u, D) \neq \emptyset \forall u \in D$.

Theorem 3.8

If $G \neq K_2$ is JTFE, then $|PN_t^f(u, D)| \geq 2, \forall u \in D$ where D is any γ_t^f -set of G .

Proof

Let D be γ_t^f -set of G . Since D is a γ_t^f -set, $|PN_t^f(u, D)| \neq \emptyset$. Assume that for some $u \in D$ $|PN_t^f(u, D)| = 1$. If there exist $w \in D$ such that $N(w) \cap D = \{u\}$ then $PN_t^f(u, D) = \{w\}$. Thus $(D - u) \cup \{y\}$ is a γ_t^f -set for any $y \in N(w)$. As $\deg(w) \geq 2$, select one $y \neq u \in N(w)$. Then D and $(D - u) \cup \{y\}$ are two distinct γ_t^f -sets of G . Containing the vertex $w_1 \rightarrow$ which contradiction to G is JTFE.

If $PN_t^f(u, D) = \{x\}$ where $x \notin D$, then select one $y \neq u \in N(x)$. Then $(D-u) \cup \{y\}$ is a γ_t^f -set. The $D \cap ((D-u) \cup \{y\}) \neq \emptyset$ which is a contradiction. Hence $|PN_t^f(u, D)| \geq 2$.

Theorem 3.9

If $G \neq mk_2$ is JTFE, then $\gamma_t^f(G) \leq \left\lceil \frac{n}{3} \right\rceil$

Proof

Assume $\gamma_t^f(G) > \frac{n}{3}$, then $d_t(G) = 2$, let $V = V_1 \cup V_2$ where V_1 and V_2 are distinct γ_t^f -sets

of G . By theorem 3.8, $|PN_t^f(u, V_1)| \geq 2$. Let $X = \{u \in V_1 / |PN_t^f(u, V_1) \cap V_1| \geq 2\}$.

$Y = \{v \in V_1 / |PN_t^f(u, V_1) \cap V_1| = \{v\} \text{ for some } u \in X\}$. And $Z = V_1 \setminus (X \cup Y)$. We assume that

$X \neq \emptyset$, for every $v \in Y$, $PN_t^f(u, V_1) \subset V_2$, for every $v \in Z$, $PN_t^f(v, V_1) \cap V_2 \neq \emptyset$. Also

$|Y| \geq 2|X|$. Thus $\left| \bigcup_{x \in V_1} (PN_t^f(x, V_1) \cap V_2) \right| \geq 2|Y| + |Z| \geq |X| + (|X| + |Y| + |Z|) > |V_1| = |V_2| = \gamma_t^f(G)$,

which is a contradiction. So $X = \emptyset$. Hence $PN_t^f(x, V_1) \cap V_2 \neq \emptyset \forall x \in V_1$. So

$|PN_t^f(x, V_1) \cap V_2| = 1, \forall x \in V_1$. We have $PN_t^f(x, V_1) \cap V_2 \neq \emptyset \forall x \in V_1$ and

$\bigcup (PN_t^f(x, V_1) \cap V_1) = V_1$. Also $PN_t^f(x, V_1) \cap V_1 = \{y\} \Leftrightarrow PN_t^f(y, V_1) \cap V_1 = \{x\}$. So

$\deg(x) = 1$ in $\langle V_1 \rangle$ for every $x \in V_1$. Hence $\deg(x) = 2$ in G , such that $\mu(xy_i) = 1$ and

$y_i \in V_2$ for $i=1,2$. Similarly $\deg(x) = 2$ in $G \forall x \in V_2$. As G is 2-regular, each component of G

is a cycle. As $G \neq mk_2$ and G is JTFE and is connected. Therefore G itself is a cycle. But

cycle C_n is not JTFE. Hence our assumption that $\gamma_t^f(G) \leq \left\lceil \frac{n}{3} \right\rceil$ is wrong. Therefore

$$\gamma_t^f(G) \leq \left\lceil \frac{n}{3} \right\rceil.$$

Theorem 4.0

If $G \neq mk_2$ is JTFE, then $\Delta(G) \leq n - 2k + 2$, where $k = \gamma_t^f(G)$

Proof

Let u be a vertex in G and let D be a γ_t^f -set for G which contains u . By thm 3.8,

$|PN_t^f(v, D)| \geq 2, \forall v \in D$. If $v_1 \neq v_2 \in D$, then $PN_t^f(v_1, D) \cap PN_t^f(v_2, D) = \emptyset$. [clearly

$PN_t^f(v_1, D) \cap PN_t^f(v_2, D) = \phi$. If $v \in PN_t^f(v_1, D) \cap D$, then $v \notin N(y), \forall y \neq v_1 \in D$ and $v \notin PN_t^f(v_1, D)$. The vertex $u \in PN_t^f(v, \delta)$ for at most one $v \in D$. So u is not adjacent to any of the vertices in $\bigcup_{v \neq u \in D} PN_t^f(v, D)$ and $\deg u \leq (n-1) - 2(|S|-1) + 1$. It is true for all $u \in V(G), \Delta \leq n - 2k + 2$.

Example:

1. C_n is not JTFE but it is total fuzzy excellent
2. $K_{m,n}$ is not JTFE (unless $m=n=1$)
3. Peterson's graph is not JTFE

Theorem 4.1

Let $G \neq K_2$ be JTFE graph. Then G contains no cut vertex.

Proof

Assume to the contrary, that G contains a cut vertex u . Then $d_t^f(G) \geq 3$ let D be the dominating set which contains u . Choose two distinct γ_t^f -sets D_2 and D_3 which is different from D_1 . Since G is JTFE, $u \notin D_2 \cup D_3$. Select two vertices v and w such that $v \in D_2 \cap N(u)$ and $w \in D_3 \cap N(u)$ and $\mu(uv) \leq \sigma(u) \wedge \sigma(v)$ and $\mu(uw) \leq \sigma(u) \wedge \sigma(w)$.

Let G_1 be the component of $G-u$ that contains v and let H_1 be subgraph of G included by $G_1 \cup \{u\}$ and let $H_2 = G - H_1$.

Case 1

Let us assume that $w \in H_1$. Then

- i) Both $D_2 \cap H_1$ and $D_3 \cap H_1$ is not dominated by any vertex of H_2
- ii) $|D_1| = |D_2| = |D_3|$
- iii) No vertex of $D_i \cap H_2, i=1,2$ is isolated in $\langle D_i \cap H_2 \rangle$
- iv) If $|D_2 \cap H_2| < |D_3 \cap H_2|$, then $|D_2 \cap H_1| > |D_3 \cap H_1|$ and $(D_2 \cap H_1) \cup (D_3 \cap H_1)$ is a total dominating set of G , which is contradiction as $|(D_2 \cap H_2) \cup (D_3 \cap H_2)| < |(D_2 \cap H_1) \cup (D_3 \cap H_1)| = |S_3| = \gamma_t^f(G)$. Similarly if $|D_3 \cap H_2| < |D_2 \cap H_2|$, we get a contradiction. Thus $|D_2 \cap H_2| = |D_3 \cap H_2|$ and

$(D_2 \cap H_1) \cup (D_3 \cap H_2)$ and D_3 are distinct γ_t^f -set of G containing $D_3 \cap H_2$. Note that $v \in D_2 \cap H_1$ dominates u , which is a contradiction the fact that G is JTFE.

Case 2

Now assume $w \in H_2$. Then $D_2 \cap H_2$ and $D_3 \cap H_1$ are total dominating sets for H_2 and $H_1 - u$, Thus $|D_2 \cap H_2| \geq \gamma_t^f(H_2)$ and $|D_3 \cap H_1| \geq \gamma_t^f(H_1 - u)$. If $|D_2 \cap H_2| > \gamma_t^f(H_2)$, then $(D_2 \cap H_1) \cup S$ is total fuzzy dominating set of G for any γ_t^f -set S of H_2 . As $|(D_2 \cap H_1) \cup S| = |D_2 \cap H_1| + |S| < |D_2 \cap H_1| + |D_2 \cap H_2| = \gamma_t^f(G)$, which we get contradiction. Similarly if $|D_3 \cap H_1| > \gamma_t^f(H_1 - u)$, $(D_3 \cap H_2) \cup S$ is total fuzzy dominating set of G , for any γ_t^f -set S of $H_1 - u$ and we get a contradiction.

Hence $|D_2 \cap H_2| = \gamma_t^f(H_2)$ and $|D_3 \cap H_1| = \gamma_t^f(H_1 - u)$. As $u, v \notin D_3$, there exist $x \in D_3 \cap H_1$, which is adjacent to v and hence c is a total fuzzy dominating set of G containing v and $(D_3 \cap H_1)$. As G is JTFE, this total fuzzy dominating set of G is not γ_t^f -set of G . Therefore $\gamma_t^f(G) \leq |D_3 \cap H_1| + |D_2 \cap H_2| = \gamma_t^f(H_1 - u) + \gamma_t^f(H_2)$.

As $D_2 \cap H_1$ and $D_2 \cap H_2$ are total fuzzy dominating set of $H_1 - u$ and respectively, $\gamma_t^f(G) = |D_2| = |D_2 \cap H_1| + |D_2 \cap H_2| \geq \gamma_t^f(H_1 - u) + \gamma_t^f(H_2) \geq \gamma_t^f(G)$.

Hence $\gamma_t^f(G) = \gamma_t^f(H_1 - u) + \gamma_t^f(H_2)$.

Now as $\gamma_t^f(G) = |D_2| = |D_2 \cap H_1| + |D_2 \cap H_2| = |D_2 \cap H_1| + \gamma_t^f(H_1 - u) + \gamma_t^f(H_2)$, we get $|D_2 \cap H_1| = \gamma_t^f(H_1 - u)$. Similarly $|D_3 \cap H_2| = \gamma_t^f(H_2)$. If $|D_1 \cap H_1| < \gamma_t^f(H_2)$, then if $S = (D_2 \cap H_1) \cup \{u\} \cup (D_1 \cap H_2)$, then S is a total fuzzy dominating set of G and as $|S| = |D_2 \cap H_1| + 1 + |D_1 \cap H_2| \leq \gamma_t^f(H_1 - u) + \gamma_t^f(H_2) - 1 = \gamma_t^f(G)$, S is a γ_t^f -set of G , again we get a contradiction as both S and D_1 contains u .

If $|D_1 \cap H_2| > \gamma_t^f(H_2)$, then $|D_1 \cap H_2| < \gamma_t^f(H_1 - u) \& (D_1 \cap H_1) \cup (D_3 \cap H_2)$ is a total fuzzy dominating set for G . [Since $u \in D_1 \cap H_1 \& w \in D_3 \cap H_2$, then $(S_1 \cap H_1) \cup (S_3 \cap H_2)$ does not contain isolated vertices].

But $|(D_1 \cap H_1) \cup (D_3 \cap H_2)| = |D_1 \cap H_1| + |D_3 \cap H_2| < \gamma_i^f(H_1 - u) + \gamma_i^f(H_2) = \gamma_i^f(G)$, which is a contradiction. Thus $|D_1 \cap H_2| = \gamma_i^f(H_2)$ & $|D_1 \cap H_1| = \gamma_i^f(H_1 - u)$ and $(D_1 \cap H_1) \cup (D_3 \cap H_2)$ is a γ_i^f -set of S which is a contradiction. Thus u is not a cut vertex.

Theorem: 4.2

Let $G \neq mK_2$ is JTFE graph. Then every vertex u is a γ_i^f -level vertex and $\gamma_i^f(G - \{u\}) = \gamma_i^f(G)$.

proof

Let $G \neq mK_2$ be JTFE graph. Let u be a vertex in G . Since G is JTFE, there exists a γ_i^f -set of G not containing u . Therefore $\gamma_i^f(G - \{u\}) \leq \gamma_i^f(G)$.

We want to prove that $\gamma_i^f(G - \{u\}) = \gamma_i^f(G)$. Let us assume that $\gamma_i^f(G - \{u\}) < \gamma_i^f(G)$. Let D be a γ_i^f -set for $G - \{u\}$. Then $D \cup \{u\}$ is a γ_i^f -set for G , for all $v \in N[u]$. Since G is connected, $N[u]$ contains at least two vertices v_1 & v_2 . Thus $D \cup \{v_1\}$ & $D \cup \{v_2\}$ are γ_i^f -set for G which is contradiction as G is JTFE. Therefore $\gamma_i^f(G - \{u\}) = \gamma_i^f(G)$.

Suppose $\gamma_i^{uf}(G, \{u\}) < \gamma_i^f(G)$, let D be a $\gamma_i^{uf}(G, \{u\})$ -set. If $u \in D$, then D is also a dominating set for G , which is a contradiction. If $u \notin D$, then D is a γ_i^f -set $G - \{u\}$ and $\gamma_i^f(G - \{u\}) < \gamma_i^f(G)$, is a contradiction.

Hence $\gamma_i^f(G - \{u\}) = \gamma_i^f(G)$.

REFERENCES

1. L.A.Zadeh, Fuzzy sets, Information and control, 1965; 8: 338-353.
2. N.Sridharan and M.Yamuna, Excellent-Just Excellent- Very excellent graphs, Journal of Math.ohy,sci, 1980; 14(5): 471-475.
3. A.Somasundaram and S.Somasundaram, Domination in Fuzzy graphs-I, Elsevier Science, 1998; 19: 787-791.
4. V.R.Kuli, Theory of domination in graphs, Vishwa internation publications, 2012.
5. Zadeh,L.A., "similarity relations and fuzzy orderingz", Information Sciences, 1971; 3(2): 177-200.

6. A.Nagoorkani and V.T.Chandrasekaran, Domination in Fuzzy graph, Advances in Fuzzy sets and systems, 2006; 1(1): 17-26.
7. E.Sampathkumar and L.Pushpalatha, Set domination in graphs , Journal of Graph theory, 1994; 18(5): 489-495.
8. T.W.Haynes, S.T.Hedetniemi and P.J.Slater, Domination in Graphs, Advanced Topics, Marcel Dekker, Inc., 1998.
9. K.R.Bhutani and A.Rosenfeld, Strong arcs in Fuzzy graphs, Information Sciences, 2003; 152: 319-322.
10. P.Debnath, Some results of domination and total domination in the direct product of two fuzzy graphs, arXiv, 1312.3444 v1 [cs.DM], 2013.