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THE ITERATION METHOD FOR STUDYING THE KLEIN-GORDON EQUATIONS

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ABSTRACT

In recently, different iterative methods viz Adomian decomposition method, variational iteration method, Homotopy Pertubation method etc. have been developed for solving linear and nonlinear ordinary and PDEs. Recently Versha and Jafery proposed an iterative method called the New Iterative Method (NIM) and successfully applied it to linear and nonlinear PDEs of integer and fractional order. In this paper we propose an efficient modification to the NIM and applied the modified NIM to obtain improved form solutions of various types of linear and nonlinear Klein-Gordon equations. The proposed modification is easy to use and we obtained excellent performance in comparison with the existing iterative methods that have been traditionally used in finding the solution of linear and nonlinear Klein-Gordon equations. The main feature of the modified NIM is that it reduces the size of calculations and gives the solution rapidly while still maintaining high degree of accuracy.

1. INTRODUCTION

The study of PDEs was started in the 18th century by Euler, d'Alembert, Lagrange and Laplace[1] as a central tool in analytical study of models in physical sciences. During this century and in early 19th century, the classical PDEs which serve as paradigms for the later development also appeared. A profusion of equations, associated with major physical phenomena, appeared during 1750-1900:

- The Euler equation of incompressible fluid flows in 1755.
- The minimal surface equation by Lagrange in 1760.

- The Monge-Ampere equation by Monge in 1775.
- The Laplace and Poisson equations by Poisson in 1813.
- The Navier Stokes equations for fluid flows in 1822-1827 by Navier.
- Maxwell's equation in electromagnetic theory in 1864.
- The Helmholtz equation and the eigenvalue problem for the Laplace operator in connection with acoustics in 1860.
- The Korteweg-De Vries equation (1896) as a model for solitary water waves.

In recent years many efficient numerical techniques have been developed for solving linear and nonlinear PDEs. Some important techniques are:

- Adomian decomposition method.[2,3]
- Modified Adomian decomposition method.[4]
- Variational iteration method.[5]
- Homotopy purterbation method.[6,7]

For an applied mathematician it becomes increasingly important to be familiar with all traditional and recently developed methods for solving PDEs, and the implementation of these methods. However, in this thesis, we will restrict our analysis to solve different models of linear and nonlinear Klein-Gordon equations.

2. PRELIMINARIES

Classification of PDEs as Homogeneous and Inhomogeneous

We can also classify a PDE as homogeneous and inhomogeneous. A PDE of any order is called homogeneous if every term of the PDE contains the dependent variable *u* or one of its derivatives; otherwise, it is called an inhomogeneous PDE.

Examples:1

$$u(x,t) = x^2 t + \frac{1}{4} L_t^{-1} u^2 x$$

The terms of the Eq. (1.1) contain partial derivatives of dependent variable only; therefore it is a homogeneous PDE.

2.1. Classification of second order PDEs

A second order linear PDE in two independent variables *x* and *y* in its general form is given by

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G \qquad [1]$$

Above equation can be classified into three basic classes which are as follows:

(i) **Parabolic**: The Eq. (1) is said to be parabolic if it satisfies the condition $B^2 - 4AC = 0$. Heat flow and diffusion process equations are the examples of Parabolic equations.

(ii) **Elliptic**: The Eq. (1) is called an elliptic equation if $B^2 - 4AC < 0$. Examples of elliptic equations are Laplace equation and Schrodinger equation.

(iii) **Hyperbolic**: The Eq. (1) is called hyperbolic if it satisfies the property $B^2 - 4AC > 0$. The examples of hyperbolic equations are wave propagation equations.

2.2. Solution of PDEs

A solution of a PDE is the value of the dependent variable, e.g. u such that it satisfies the PDE under discussion and satisfies the given conditions as well.

2.3. Initial Value Problem

The PDE in which all the conditions for finding constants of integration are given only on the starting point is called an initial value problem.

3. Boundary Value Problem

A PDE that controls the mathematical behavior of physical phenomenon in a bounded domain D and the dependent variable u is prescribed at the boundary of domain is called the boundary value problem.

Next we give a detailed explanation of some important numerical methods that are being used now a days in solving linear and nonlinear PDEs in different branches of applied Mathematics, engineering and Physics. The first method to be discussed is the well-known *Adomian decomposition method.*

4. Adomian Decomposition Method

The *Adomian decomposition method (ADM)* has been receiving much attention of Mathematicians in recent years in applied Mathematics in general and in the area of series solutions in particular. The decomposition method demonstrates fast convergence of the solution and attacks to homogeneous and inhomogeneous problems in a direct and straightforward fashion without using linearization, perturbation or any other restrictive assumption that may change the physical behavior of the model under consideration. The ADM was introduced and developed by George Adomian,[3,4] chair of the center of applied mathematics at the University of Georgia and is well addressed in the literature. This method has proved to be a competitive alternative to the Taylor series method and other series techniques. The ADM consists of decomposing the unknown function $u(x, y)$ of any equation into a sum of an infinite number of components defined by the decomposition series

$$u(x,y)=\sum_{n=0}^{\infty}u_n(x,y) \qquad [2]$$

Where the components $u_n(x,y)$, $n \geq 0$ are to determine in a recursive manner. Although the linear term u is expressed as an infinite series of components, the ADM requires a special representation for the nonlinear terms such as $u^2, u^3, u^4, \sin(u), e^u, uu_x, u^2{}_x$ etc. that appear in the equation. . To get a clear overview of ADM, we first consider the linear differential equation written in an operator form by

$$Lu+Ru=g, \qquad [3]$$

Where L is, mostly, the lower order derivative which is assumed to be invertible, R is other linear differential operator and g is source term. Applying L^{-1} to both sides of Eq. (3) and using the given conditions, we obtain

$$u=f-L^{-1}(Ru), \qquad [4]$$

The ADM defines the solution u by an infinite series of components given by

$$u=\sum_{n=0}^{\infty}u_n, \qquad [5]$$

Substituting Eq. (4) in Eq. (5) leads to

$$\sum_{n=0}^{\infty}u_n=f-L^{-1}(R(\sum_{n=0}^{\infty}u_n)) \qquad [6]$$

Alternatively, Eq. (6) can be written as

$$u_0+u_1+u_2+u_3+\ldots=f-L^{-1}(R(u_0+u_1+u_2+u_3+\ldots)) \qquad [7]$$

The components u_0, $u1$, $u2$, $\cdots$ are now determined by the following relation

$$u_0=f,$$
$$u_{k+1}=-L^{-1}(R(u_k)), \qquad k\geq 0. \qquad [8]$$

Having determined these components, we then substitute it into Eq. (1.6) to obtain the solution in a series form. We demonstrate the ADM by applying it to linear PDEs.

Example 2. *Using ADM, solve the homogeneous linear Klein-Gordon equation*

$$u_{tt} - u_{xx} + u = 0, \qquad [9]$$

With initial conditions

$$u(x,0) = 0,\ u_t(x,0) = x. \qquad [10]$$

Solution: Applying L_t^{-1} on both sides of Eq. (9) and using the decomposition series for $u(x,t)$ we get

$$\sum_{n=0}^{\infty} u_n = xt + L_t^{-1}((\sum_{n=0}^{\infty} u_n(x,t))_{xx} - \sum_{n=0}^{\infty} u_n(x,t)), \qquad [11]$$

Close examination of Eq. (11) suggests that the recursive relation is

$$u_0(x,t) = xt,$$
$$u_{k+1} = L_t(u_{kxx}(x,t) - u_k(x,t)), \quad k \geq 0 \qquad [12]$$

That in turn gives

$$u_0(x,t) = xt,$$
$$u_1 = L_t^{-1}(u_{0xx}(x,t) - u_0(x,t) = -\frac{1}{3!}xt^3$$
$$u_2 = L_t^{-1}(u_{1xx}(x,t) - u_1(x,t) = \frac{1}{5!}xt^5, \qquad [13]$$

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In view of Eq. (13) the series solution is

$$u(x,t) = x(t - \frac{1}{3!}xt^3 + \frac{1}{5!}xt^5.......) \qquad [14]$$

So that the exact solution is given by $u(x, t) = x\sin(t)$.

5. Applications of ADM to Nonlinear PDEs

The nonlinear PDEs arise in different areas of Physics, engineering, and applied Mathematics. In this section, we discuss the implementation of ADM to nonlinear PDEs. For this purpose, we consider the nonlinear partial differential equation in an operator form:

$$L_x u(x,y) + L_y u(x,y) + R(u(x,y) + F(u(x,y))) = g(x,y), \qquad [15]$$

The use of the operators L_x and L_y depends upon two facts:

(i) The operator of lowest order should be selected to minimize the size of computational work.

(ii) The selected operator of lowest order should be of best known conditions to accelerate the evaluation of the components of the solution.

If we assume that the operator L_x meets the above two bases of selection, then we get

$$L_x u(x,y) = g(x,y) - L_y u(x,y) - R(u(x,y) - F(u(x,y))) \qquad [16]$$

Applying the inverse operator $L^{-}{}_{x}{}^{1}$ on both sides of Eq. (16) gives

$$u(x,y) = \sum_{n=0}^{\infty} u_n(x,y), \qquad [17]$$

$$u(x,y) = \phi_0 + L_x^{-1} g(x,y) - L_x^{-1} L_y u(x,y) - L_x^{-1} R(u(x,y)) - L_x^{-1} F(u(x,y)) \qquad [18]$$

Where

$$\phi_0 = \begin{cases} u(0,y \\ u(0,y) + x u_x(0,y \\ u(0,y) + x u_x(0,y) + \frac{1}{2!} x^2 u_{xx}(0,y \\ u(0,y) + x u_x(0,y) + \frac{1}{2!} x^2 u_{xx}(0,y) + \frac{1}{3!} x^3 u_{xxx}(0,y \\ . \\ . \\ . \end{cases}$$

Proceeding in the same manner we get the series solution

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) \qquad [19]$$

And the nonlinear term $F(u(x,y))$ is given by

$$F(u(x,y)) = \sum_{n=0}^{\infty} A_n, \qquad [20]$$

Example 3. *Using ADM, solve the nonlinear PDE*

$$u_t = x^2 + \frac{1}{4}u^2 x, \qquad [21]$$

$$with\,initial\,condition$$

$$u(x,0) = 0, \qquad [22]$$

$$Where \quad u = u(x,t).$$

Solution: Operating $L^{-}{}_{t}{}^{1}$ to Eq.(21), we have

$$u(x,t) = x^2 t + \frac{1}{4}L_t^{-1}u^2 x \qquad [23]$$

In series solution $u(x,t)$ can be written as

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) \qquad [24]$$

and the nonlinear term u_x^2 is defined as

$$u_x{}^2 = \sum_{n=0}^{\infty} A_n, \qquad [25]$$

Where A_n are Adomian polynomials for $n \geq 0$. Now using Eqs. (25) and (24) in Eq. (23), we obtain

$$\sum_{n=0}^{\infty} u_n(x,t) = x^2 t + \frac{1}{4}L_t^{-1}\left(\sum_{n=0}^{\infty} A_n\right) \qquad [26]$$

From this we get the following relation

$$u_0(x,t) = x^2 t$$

$$u_{k+1}(x,t) = \frac{1}{4}L_t^{-1}(A_k), \quad k \geq 0.$$

Furthermore, the Adomian polynomials A_n are given by

$$A_0 = u20x,$$

$$A_1 = 2u0xu1x$$

$$A_2 = 2u0xu2x + u21x,$$

$$\ldots$$

And so on. The first few components are given by

$$u_0(x,t) = x^2 t,$$

$$u_1(x,t) = \frac{1}{3}x^2 t^3,$$

$$u_2(x,t) = \frac{2}{15}x^2 t^5$$

$$\ldots$$

The series solution by combining the above components is given as

$$u(x,t) = x^2\left(t + \frac{1}{3}t^3 + \frac{2}{15}t^5 + ...\right) \qquad [27]$$

And in closed form

$$u(x,t) = x^2 \tan t \qquad [28]$$

5.1. Modified Decomposition Method (MDM)

To accelerate the convergence of the series solution Wazwaz[5] presented a modification of the ADM. The modified decomposition method (MDM) can be applied, wherever it is appropriate, to all PDEs of any order. To explain the technique, we consider the PDE in an operator form

$$Lu + Ru = g$$

where L is the highest order derivative, R is a linear differential operator of less order or equal order to L, and g is the source term.

we obtain

$$u = f - L^{-1}(\mathrm{Ru}) \qquad (29)$$

where f represents the terms arising from the given initial condition and from integrating the source term g. Now the series solution can be given as

$$u = \sum_{n=0}^{\infty} u_n \qquad (30)$$

The ADM admits the use of the recursive relation

$$u_0 = f \qquad (31)$$

$$u_{k+1} = L^{-1}(\mathrm{R}(\mathrm{u}_k)), \mathrm{k} \geq 0$$

The MDM introduces a minor modification to Eq. (31) so that the determination of the components of u can be made faster and easier. For specific cases, the function f can be decomposed as sum of two functions, namely f_1 and f_2. In other words, we can set

$$f = f_1 + f_2 \qquad (32)$$

To reduce the size of calculations, we identify the zeroth component u_0 by one part of f, namely f_1 or f_2. The other part of f can be added to the component u_1 among other terms. Consequently, the modified recursive relation can be expressed as

$$u_0 = f,$$
$$u_1 = f_2 - L^{-1}(\mathrm{R}(\mathrm{u}_0)),$$
$$u_{k+1} = L^{-1}(\mathrm{R}(\mathrm{u}_k)), \mathrm{k} \ge 0.$$

Two important remarks related to the above modified technique can be made here. First, by proper selection of the functions f_1 and f_2, the exact solution u may be obtained by using very few iterations, and sometimes by evaluating only two components. The success of this modification depends only on the choice of f_1 and f_2, and this can be made through trials. Secondly, if f consists of one term only, the standard ADM should be applied. In addition to MDM, to accelerate the convergence of the solution in ADM, a technique called noise terms phenomenon can also be employed which is explained next.

5.2. The Noise Terms Phenomenon

The main ideas about noise terms and the noise terms phenomenon can be outlined as follows:

(i) The identical terms with opposite signs that arise in the components u_0 and u_1 are called noise terms. These identical terms with opposite signs may exist only for inhomogeneous differential equations.

(ii) By canceling the noise terms appearing between u_0 and u_1, even though u_1 contains more terms, the remaining non-canceled terms of u_0 may give the exact solution of the PDE. Therefore, it is essential to check that the non-canceled terms of u_0 satisfy the given PDE. Moreover, if the non-canceled terms of u_0 did not satisfy the given PDE, or the noise terms did not appear between u_0 and u_1, then it is required to evaluate more components of u to find the solution in a series form.

(iii) The conclusion about the self-canceling noise terms was based on observations drawn from solving specific models without giving any proof. The interested reader is referred to[11,12] for further readings about the noise terms phenomenon.

(iv) Wazwaz concluded in[12,13] that the zeroth component u_0 must contain the exact solution u among other terms. Also, the nonhomogenity condition does not always guarantee the appearance of the noise terms as examined by Wazwaz.

Next, we demonstrate the applications of MDM and the noise terms phenomenon with the help of different models.

5.3. Applications of Noise Terms Phenomenon and MDM

Example 4. *Using MDM, solve the nonlinear PDE,*

$$u_t + uu_x = x + xt^2, \quad (33)$$

with initial conditions

$$u(x,0) = 0, t > 0, \quad (34)$$

where $u = u(x,t)$.

Solution. The operator form of Eq. (33) is

$$L_t u(x,t) = x + xt^2 - uu_x \quad (35)$$

Applying L_t^{-1} on both sides of Eq. (35) and using the initial conditions, we get

$$u(x,t) = xt + \frac{1}{3}xt^3 - L_t^{-1}uu_x \quad (36)$$

Using the decomposition assumptions for the linear term $u(x,t)$ and for the nonlinear term uu_x defined by

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t)$$

and

$$uu_x = \sum_{n=0}^{\infty} A_n$$

into (36) gives

$$\sum_{n=0}^{\infty} u_n(x,t) = xt + \frac{1}{3}xt^3 - L_t^{-1}(\sum_{n=0}^{\infty} A_n). \quad (37)$$

This gives the recursive relation admitting MDM

$$\begin{aligned} u_0(x,t) &= xt \\ u_1 &= \frac{1}{3}xt^3 - L_t^{-1}(A_0) \\ u_{k+2} &= -L_t^{-1}(A_{k+1}), \quad k \geq 0 \end{aligned} \quad (38)$$

Finally, we obtained

$$\begin{aligned} u_0(x,t) &= xt \\ u_1 &= \frac{1}{3}xt^3 - L_t^{-1}(xt^2), \\ u_{k+2} &= 0, \quad k \geq 0 \end{aligned} \quad (39)$$

In view of (39), the exact solution is given by

$$u(x, y) = xt.$$

Example 5. *Use the noise terms phenomenon and MDM to solve the following inhomogeneous nonlinear Klein-Gordon equation*

$$u_{tt} - u_{xx} + u_2 = 2x_2 - 2t_2 + x_4 t_4 \qquad (40)$$

with initial conditions

$$u(x,0) = 0, u_t(x,0) = 0. \qquad (41)$$

Solution. Applying L^{-1}_{t} on both sides of Eq. (40) and using the initial conditions we get

$$u_0(x,t) = x^2 t^2 - \frac{1}{6}t^4 + \frac{1}{30}x^4 t^6 \quad ,$$

$$u_{k+1} = L_t^{-1}(u_{kxx}) - L_t^{-1}(A_k) \text{ , } k \geq 0$$

using the above relation, we get the following approximations

$$u_0(x,t) = x^2 t^2 - \frac{1}{6}t^4 + \frac{1}{30}x^4 t^6$$

$$u_{k+1} = L_t^{-1}(u_{0xx}(x,t)) - L_t^{-1}(A_0) = \frac{1}{6}t^4 - \frac{1}{30}x^4 t^6 \ldots\ldots$$

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Canceling the noise terms in $u_0(x,t)$ that appears in $u_1(x,t)$ and verifying that the remaining term satisfies the equation leads to the exact solution

$$u(x,t) = x^2 t^2 \qquad (42)$$

Now we apply MDM to demonstrate the fast convergence. The MDM introduces the relation

$$u_0(x,t) = x^2 t^2$$

$$u_1 = -\frac{1}{6}t^4 + \frac{1}{30}x^4 t^6 + L_t^{-1}(u_{0xx}(x,t) - A_0) = 0, \qquad (43)$$

$$u_{k+1} = 0, \ k \geq 0.$$

Finally, we get the series solution

$$u_0(x,t) = \sum_{n=0}^{\infty} u_n(x,t) = x^2 t^2 \qquad (44)$$

which is the exact solution.

Example 6. *Use the noise terms phenomenon and the MDM to solve the inhomogeneous nonlinear Klein-Gordon equation*

$$u_{tt} - u_{xx} + u^2 = x^2t^2, \qquad (45)$$

with initial conditions

$$u(x,0) = 0, u_t(x,0) = x. \qquad (46)$$

Solution. Applying $L^{-}{}_{t}{}^{1}$ on both sides of Eq. (45) and using the initial conditions, we get

$$u_0(\mathrm{x},\mathrm{t}) = \mathrm{x}\,t + \frac{1}{12}x^2t^4,$$

$$u_{k+1} = L_t^{-1}(\mathrm{u}_{kxx}) - \mathrm{L}_t^{-1}(\mathrm{A}_k)\ ,\ \mathrm{k} \geq 0,$$

using the above relation, we get the following approximations

$$u_0(\mathrm{x},\mathrm{t}) = \mathrm{x}\,t + \frac{1}{12}x^2t^4,$$

$$u_{k+1} = L_t^{-1}(\mathrm{u}_{kxx}) - \mathrm{L}_t^{-1}(\mathrm{A}_k) = \frac{1}{80}t^6 - \frac{1}{12}x^2t^4 - \frac{1}{252}x^3t^7 + \frac{1}{12960}x^4t^{10}.$$

Canceling the noise term, $\frac{1}{12}x^2t^4$ that appears in $u_0 u_0(x,t)$, and verifying that the remaining non-canceled terms satisfy the given equation, the exact solution can be obtained as

$$u(x,t) = xt.$$

Next the modified recursive relation can be rewritten as

$$u_0(\mathrm{x},\mathrm{t}) = \mathrm{x}^2\,t^2$$

$$u_1 = \frac{1}{12}x^2t^4 + L_t^{-1}(\mathrm{u}_{0xx}(\mathrm{x},\mathrm{t}) - \mathrm{A}_0) = 0,$$

$$u_{k+1} = 0,\ k \geq 0.$$

Finally, we get the exact solution

$$u(x,t) = xt.$$

5.4. The Variational Iteration Method

In recent years, the variational iteration method (VIM) established by Ji-Huan He [14] has been thoroughly investigated by many researchers to handle a wide range of scientific and engineering applications: linear, nonlinear, homogeneous and inhomogeneous. This method is effective and reliable for analytic and numerical treatment of the models. The method gives rapidly convergent successive approximations of the exact solution if such a solution exists. The VIM deals nonlinear terms in a straightforward manner without any specific assumptions as in ADM, perturbation techniques, etc. In what follows, we give a brief description of the

VIM by considering the following differential equation in operator form

$Lu + Nu = g(t),$

where L and N are linear and nonlinear operators respectively and $g(t)$ is the source inhomogeneous term. The VIM presents a correction functional for Eq. above in the form

$$u_{n+1}(t) = u_n(t) + \int_0^t (\lambda(\xi)(Lu_n(\xi) + N\tilde{u}_n(\xi) - g(\xi))\,d\xi,$$

where λ is a general Lagrange multiplier, which can be identified optimally via the variational theory, and $\tilde{u}_n$ is a restricted variation which means $\delta\tilde{u}_n = 0$. The VIM requires the determination of the Lagrange multiplier $\lambda(\xi)$ that will be identified optimally by using Integration by parts usually so that we can use

$$\int \lambda(\xi)u'_n(\xi)d\xi = \lambda(\xi)u_n(\xi) - \int \lambda'(\xi)u_n(\xi)d\xi$$

$$\int \lambda(\xi)u''_n(\xi)d\xi = \lambda(\xi)u'_n(\xi) - \lambda'(\xi)u_n(\xi) + \int \lambda''(\xi)u_n(\xi)d\xi$$

and so on. Having determined the Lagrange multiplier $\lambda(\xi)$, the successive approximations $u_{n+1},\ n \geq 0,$ of the solution u will be readily obtained upon using any selective function u_o. Consequently the solution $u = \lim_{n\to\infty} u_n$. Next, we demonstrate the use of VIM by applying it to various models.

5.5. Applications of VIM

Example 7. *Using VIM, solve the homogeneous linear Klein-Gordon equation*

$$u_{tt} - u_{xx} + u = 0, \tag{47}$$

with initial conditions

$$u(x,0) = 0,\ u_t(x,0) = x. \tag{48}$$

Solution. The correction functional for this equation reads

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(\xi)\left(\frac{\partial^2 u_n(x,\xi)}{\partial \xi^2} - \frac{\partial^2 \tilde{u}_n(x,\xi)}{\partial x^2} + \tilde{u}_n(x,\xi)\right)d\xi.$$

This yields the stationary conditions

$$1 - \lambda_0|_{\xi=t} = 0,$$
$$\lambda|_{\xi=t} = 0,$$
$$\lambda_{00}|_{\xi=t} = 0,$$

from above conditions we get

$$\lambda = \xi - t. \tag{49}$$

Substituting this value of the Lagrange multiplier into the correction functional gives the iteration formula

$$u_{n+1}(\mathrm{x,t}) = \mathrm{u}_n(\mathrm{x,t}) + \int_0^t (\xi - t)\left(\frac{\partial^2 \mathrm{u}_n(\mathrm{x},\xi)}{\partial \xi^2} - \frac{\partial^2 u_n(\mathrm{x},\xi)}{\partial x^2} + u_n(\mathrm{x},\xi)\right) \mathrm{d}\,\xi. \qquad (50)$$

Considering the given initial values, we can select $u_0(x,t) = xt$. Using this selection into Eq. (50), we obtain the following successive approximations

$u_0(\mathrm{x,t}) = \mathrm{xt},$

$u_1(\mathrm{x,t}) = \mathrm{xt} - \frac{1}{3!}xt^3,$

$u_2(\mathrm{x,t}) = \mathrm{xt} - \frac{1}{3!}xt^3 + \frac{1}{5!}xt^5,$

$u_3(\mathrm{x,t}) = \mathrm{xt} - \frac{1}{3!}xt^3 + \frac{1}{5!}xt^5 - \frac{1}{7!}xt^7,$

.

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.

$u_n(\mathrm{x,t}) = \mathrm{x}(\mathrm{t} - \frac{1}{3!}t^3 + \frac{1}{5!}t^5 - \frac{1}{7!}t^7 + \ldots).$

Hence the exact solution is given by $u(x,t) = x\sin(t)$.

6. METHODOLOGY

The Modified New Iterative Method for solving Klein-Gordon Equations

6.1. Introduction

Klein-Gordon equation is one of the most important mathematical model in quantum field theory, nonlinear optics and plasma physics. The Klein-Gordon equation appears in physics in linear and nonlinear forms. This equation has been extensively studied by using traditional methods, such as finite difference method, finite element method and collocation method. Backlund transformations and the inverse scattering method were also applied to handle this equation. The methods investigated the concepts of existence, uniqueness of the solution and the weak solution as well. The objectives of these studies were mostly focused on the determination of numerical solutions where a considerable volume of calculations is usually needed.

In [3, 6, 11, 13, 14, 16, 17, 18, 19, 20], ADM, VIM and homotpoy perturbation method were applied to obtain the exact solutions of linear and nonlinear Klein-Gordon equation. In this thesis, we apply NIM developed by Versha[9] to Klein-Gordon equation.

The NIM converges for homogeneous Klein-Gordon equation but does not always converge for linear and nonlinear inhomogeneous Klein-Gordon equations. We propose an efficient modification to NIM to apply it to both linear and nonlinear inhomogeneous Klein-Gordon equation. The modification is slight but the obtained results show that the modified technique is practical and efficient. Moreover, the modified technique minimizes the amount of calculations as compared to ADM, VIM and HPM.

6.2. Modified New Iterative Method(*MNIM*)

The modified new iterative method(MNIM) is based on including particular terms of the source term of inhomogeneous Klein-Gordon equation into the integral representing $N(u)$ in NIM.[9] This selection is based on the following rules:

If the source term is function of the independent variable, x only, we include it in $N(u)$.

1. If the source term is function of both independent variables x and t, we include it in $N(u)$.
2. If the source term contains the terms which are functions of x,t and both x and t, then we include in $N(u)$ the terms involving t and both x and t.
3. If the source term is then NIM[9] can be applied to obtain the solution.

Next we demonstrate MNIM by applying it to various models of Klein-Gordon equation.

6.3. Applications of NIM to homogeneous linear Klein-Gordon equations

Here the NIM[9] is applied to homogeneous and inhomogeneous Klein-Gordon equations. The obtained results show the excellent performance of the method.

6.4. Applications of MNIM to inhomogeneous Klein-Gordon equations

Here we apply the NIM[9] to linear and nonlinear inhomogeneous Klein-Gordon equation and demonstrate that it does not always converge to exact solution. The Modified new iterative method is then applied to find the exact solutions where NIM is not applicable.

Example 6.1. *Using MNIM, solve the inhomogeneous nonlinear Klein-Gordon equation*

$$u_{tt} - u_{xx} + u^2 = x^2 t^2, \tag{51}$$

with initial conditions

$$u(x,0) = 0,\; u_t(x,0) = x. \tag{52}$$

Solution. The exact solution of Eq. (51) is

$$u(x,\mathrm{t}) = \sum_{0}^{\infty} u_i = xt.$$

(a) **New Iterative Method**

Integrating Eq. (51) from 0 to t twice we get

$$u = xt + \frac{x^2t^4}{12} + \int_0^t\int_0^t (u_{xx} - u^2)dtdt.$$

$$set\ u_0 = xt + \frac{x^2t^4}{12}\ and\ N(u) = \int_0^t\int_0^t (u_{xx} - u^2)dtdt$$

Now the successive approximations are

$$u_0 = xt + \frac{x^2t^4}{12}$$

$$u_1 = N(u_0) = \int_0^t\int_0^t (\frac{d^2}{dx^2}(u_0) - u_0{}^2)dtdt = \int_0^t\int_0^t (\frac{t^4}{6} - \frac{1}{144}t^8x^4 - \frac{1}{6}t^5x^3 - t^2x^2)dtdt =$$
$$-\frac{1}{12960}t^{10}x^4 - \frac{1}{252}t^7x^3 + \frac{1}{180}t^6 - \frac{1}{12}t^4x^2$$

$$u_0 + u_1 = xt + \frac{x^2t^4}{12} - \frac{1}{12960}t^{10}x^4 - \frac{1}{252}t^7x^3 + \frac{1}{180}t^6 - \frac{1}{12}t^4x^2 = -\frac{1}{12960}t^{10}x^4$$
$$-\frac{1}{252}t^7x^3 + \frac{1}{180}t^6 + xt$$

$$N(u_0 + u_1) = \int_0^t\int_0^t (\frac{d^2}{dx^2}(u_0 + u_1) - (u_0 + u_1)^2)dtdt = \int_0^t\int_0^t ((-\frac{1}{1080}t^{10}x^2 - \frac{1}{42}t^7x) -$$
$$(\frac{1}{167961600}t^{20}x^8 + \frac{1}{1632960}t^{17}x^7 - \frac{1}{1166400}t^{16}x^4 + \frac{1}{63504}t^{14}x^6 - \frac{1}{22680}t^{13}x^3 +$$
$$\frac{1}{32400}t^{12} - \frac{1}{6480}t^{11}x^5 - \frac{1}{126}t^8x^4 + \frac{1}{90}t^7x + t^2x^2))dtdt = -\frac{1}{77598259200}t^{22}x^8 -$$
$$\frac{1}{55847320}t^{19}x^7 + \frac{1}{356918400}t^{18}x^4 - \frac{1}{15240960}t^{16}x^6 + \frac{1}{4762800}t^{15}x^3 + \frac{1}{5896800}t^{14} +$$
$$\frac{1}{1010880}t^{13}x^5 - \frac{1}{142560}t^{12}x^2 + \frac{1}{11340}t^{10}x^4 - \frac{11}{22680}t^9x - \frac{1}{12}t^4x^2,$$

$$u_2 = N(u_0 + u_1) - N(u_0) = (-\frac{1}{77598259200}t^{22}x^8 - \frac{1}{55847320}t^{19}x^7 + \frac{1}{356918400}t^{18}x^4 -$$
$$\frac{1}{15240960}t^{16}x^6 + \frac{1}{4762800}t^{15}x^3 + \frac{1}{5896800}t^{14} + \frac{1}{1010880}t^{13}x^5 - \frac{1}{142560}t^{12}x^2 + \frac{1}{11340}t^{10}x^4 -$$
$$\frac{11}{22680}t^9x - \frac{1}{12}t^4) - (-\frac{1}{12960}t^{10}x^4 - \frac{1}{252}t^7x^3 + \frac{1}{180}t^6 - \frac{1}{12}t^4x^2) = -\frac{1}{77598259200}t^{22}x^8 -$$
$$\frac{1}{55847320}t^{19}x^7 + \frac{1}{356918400}t^{18}x^4 - \frac{1}{15240960}t^{16}x^6 + \frac{1}{4762800}t^{15}x^3 + \frac{1}{5896800}t^{14} - \frac{1}{5896800}t^{14} +$$
$$\frac{1}{1010880}t^{13}x^5 - \frac{1}{142560}t^{12}x^2 + \frac{1}{11340}t^{10}x^4 - \frac{11}{22680}t^9x + tx,$$

which shows that iterations are diverging.

a. **Modified New iterative Method**

Integrating Eq. (51) from 0 to *t* twice we get

$$u = xt + \int_0^t \int_0^t (u_{xx} - u^2 + x^2t^2) dtdt.$$

$$set\ u_0 = xt\ and\ N(u) = \int_0^t \int_0^t (u_{xx} - u^2 + x^2t^2) dtdt$$

Now the successive approximations are

$$u_0 = xt,$$

$$u_1 = N(u_0) = \int_0^t \int_0^t ((u_0)_{xx} - u_0{}^2 + x^2t^2) dtdt = \int_0^t \int_0^t (0 - x^2t^2 + x^2t^2) dtdt = 0$$

$$u_0 + u_1 = xt,$$

$$N(u_0 + u_1) = \int_0^t \int_0^t ((u_0 + u_1)_{xx} - (u_0 + u_1)^2 + x^2t^2) dtdt = 0$$

$$u_2 = N(u_0 + u_1) - N(u_0) = 0$$

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Hence the series solution is

$$u(x,t) = \sum_0^\infty u_i = xt$$

which is exact solution.

Example *Using MNIM, solve the following inhomogeneous linear Klein-Gordon equation*

$$u_{tt} - u_{xx} - u = -cos(x)cos(t), \tag{52}$$

with initial conditions

$$u(x,0) = cos(x),\ u_t(x,0) = 0 \tag{53}$$

Solution. Integrating Eq. (52) from 0 to *t* twice we get

$$u = cos(x)cos(t) + \int_0^t \int_0^t (u_{xx} + u) dtdt.$$

$$set\ u_0 = cos(x)cos(t)\ and\ N(u) = \int_0^t \int_0^t (u_{xx} + u) dtdt$$

Now the successive approximations are

$$u_0 = cos(x)cos(t)$$
$$u_1 = N(u_0) = \int_0^t\int_0^t ((u_0)_{xx} + u_0)dtdt = \int_0^t\int_0^t (-cos(x)cos(t) + cos(x)cos(t))dtdt = 0$$
$$u_0 + u_1 = cos(x)cos(t)$$
$$u_2 = N(u_0 + u_1) - N(u_0) = 0$$

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Hence

$$u(x,t) = \sum_0^\infty u_i = cos(x)cos(t)$$

which is exact solution.

Example. *Using MNIM, solve the following inhomogeneous linear Klein-Gordon equation*

$$u_{tt} - u_{xx} - u = -cos(x)\sin(t), \qquad (54)$$

with initial conditions

$$u(x,0) = 0, u_t(x,0) = cos(x). \qquad (55)$$

Solution. Integrating equation (54) from 0 to *t* twice we get

$$u = cos(x)\sin(t) + \int_0^t\int_0^t (u_{xx} + u)dtdt.$$

$$set\ u_0 = cos(x)\sin(t)\ and\ N(u) = \int_0^t\int_0^t (u_{xx} + u)dtdt$$

Now the successive approximations are

$$u_0 = cos(x)\sin(t)$$
$$u_1 = N(u_0) = \int_0^t\int_0^t ((u_0)_{xx} + u_0)dtdt = \int_0^t\int_0^t (-cos(x)\sin(t) + cos(x)\sin(t))dtdt = 0$$
$$u_0 + u_1 = cos(x)\sin(t)$$
$$u_2 = N(u_0 + u_1) - N(u_0) = 0$$

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Hence the series solution is

$$u(x,t) = \sum_0^\infty u_i = cos(x)\cos(t)$$

which is exact solution.

RESULTS AND DISCUSSION

In this paper we have made an modification to NIM to apply it to various linear and nonlinear models of Klein-Gordon equations. The modified technique is slight but easy to use and have many advantages over the existing methods that have been used for solving Klein-Gordon equations. The solutions were obtained in a direct and straightforward manner without any assumptions or transformations that may change the physical behavior of the problem by using MNIM. The obtained results show that the MNIM reduces the size of calculations. The method has been compared with the exact solutions to assess the efficiency of the MNIM. While the Adomian decomposition method requires the determination of tedious Adomian polynomials, and the variational iteration method requires the determination of the Lagrange multiplier, the MNIM is independent of any such type of constraints. The method is validated by applying it to several physical models of Klein-Gordon equations. In MNIM a few approximations can be used to achieve a high degree of accuracy. We have found that MNIM is an efficient way to approach the exact solution of Klein-Gordon equations.

Future Work

(i) Apply NIM to linear and nonlinear boundary value problems.

(ii) Apply MNIM to the physical problems where NIM is not applicable.

(iii) Extend NIM to singular boundary value problems.

(iv) Extend NIM to apply it to system of partial differential equations.

(v) Use NIM to find solitary wave solutions of different physical models.

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