

FRACTIONAL HARDY-TYPE INEQUALITIES

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ABSTRACT

Since the last few decades, the addition of abundant literature on Hardy-type inequalities proves its importance in the field (see [1], [2], [3], [4], [5], [6], [7], [8], [9], [10], [11]). We give some Hardy-type inequalities for

fractional integrals and fractional derivatives in the second chapter. The main objective is to review the paper by Sajid Iqbal et. al.^[12] In this article we mainly focused on the mean value theorem for Hadamard-type fractional integral and the related means for fractional integral and derivatives.

Convex Function

Definition 1.1.1

A function $\Phi : I \rightarrow R$ is called a convex function in the Jensen sense if,

$$\Phi\left(\frac{s+t}{2}\right) \leq \frac{\Phi(s) + \Phi(t)}{2} \quad (1.1.1)$$

$\forall s, t \in I$. A J-convex function Φ is said to be strictly J-convex sense if \forall pairs of points (s, t) , $s \neq t$ strict inequality holds in (1.1.1).

Theorem 1.1.2.

Let $\Phi : I \rightarrow R$ be a function on the interval $I \subseteq \mathbb{R}$ such that Φ'' exists on I . Then Φ is convex iff $\Phi''(y) \geq 0$ And if $\Phi''(y) > 0$ on I , then Φ is strictly convex on the interval.

Lemma 1.1.3.

Let $f \in C^2(I)$ and let I be a compact interval such that,

$$m \leq f''(y) \leq M, \forall y \in I.$$

Consider two functions Φ_1, Φ_2 , defined as

$$\Phi_1(y) = \frac{My^2}{2} - f(y),$$

$$\Phi_2(y) = f(y) - \frac{my^2}{2}$$

Then Φ_1 and Φ_2 are convex on I .

Jensen's Inequality

Jensen's inequality was named after the Danish mathematician Johan Jensen who proved it in 1906.^[14] In the discrete form, Jensen's inequality asserts that for a function Φ

$$\Phi\left(\frac{1}{S_n} \sum_{i=1}^n s_i t_i\right) \leq \frac{1}{S_n} \sum_{i=1}^n s_i \Phi(t_i) \quad (1.2.1)$$

holds, where Φ is convex function on the interval $I \subseteq \mathbb{R}$, where s_i are positive real numbers

and $t_i \in I (i = 1, 2, \dots, n)$, while $S_n = \sum_{i=1}^n s_i$.

The inequality (1.2.1) is strict if Φ is strictly convex except when $t_1 = t_2 = \dots = t_n$.

Theorem 1.2.2

Let (Ω, A, μ) be a probability space $-\infty \leq a < b \leq \infty$ and if $f, \Phi \circ f \in L^1(\mu)$ be such that $a \leq f(x) \leq b$ for all $t \in \Omega$. Then the inequality

$$\Phi\left(\int_{\Omega} f(x) d\mu(x)\right) \leq \int_{\Omega} \Phi(f(x)) d\mu(x) \quad (1.2.2)$$

holds for any convex function $\Phi: [a, b] \rightarrow \mathbb{R}$.

AM-GM Inequality

If a_1, a_2, \dots, a_n are non-negative and real numbers and $\lambda_1, \lambda_2, \dots, \lambda_n$ are non-negative and real numbers such that $\lambda_1 + \lambda_2 + \dots + \lambda_n = 1$.^[13]

$$\prod_{k=1}^n a_k^{\lambda_k} + \prod_{k=1}^n b_k^{\lambda_k} \leq \prod_{k=1}^n (a_k + b_k)^{\lambda_k}$$

Weighted AM-GM inequality

$$\prod_{k=1}^n \left(\frac{a_k}{a_k + b_k}\right)^{\lambda_k} \leq \sum_{k=1}^n \lambda_k \left(\frac{a_k}{a_k + b_k}\right)$$

Similarly

$$\prod_{k=1}^n \left(\frac{ak}{ak+bk} \right)^{\lambda_k} \leq \sum_{k=1}^n \lambda_k \left(\frac{ak}{ak+bk} \right)$$

Summing up

$$\prod_{k=1}^n \left(\frac{1}{(ak+bk)^{\lambda_k}} \right) \left[\prod_{k=1}^n a_k^{\lambda_k} + \prod_{k=1}^n b_k^{\lambda_k} \right] \leq \sum_{k=1}^n \lambda_k = 1$$

If a, b, c are the lengths of sides of a triangle and

$$2s = a + b + c$$

$$\frac{a^n}{b+c} + \frac{b^n}{c+a} + \frac{c^n}{a+b} \geq \left(\frac{2}{3} \right)^{n-2} S^{n-1}, n \geq 1$$

when $n = 1$, result equal to Nesbitt Inequality for $n \geq 2$

$$\begin{aligned} \frac{a^n}{b+c} + \frac{b^n}{c+a} + \frac{c^n}{a+b} &\geq \frac{(a+b+c)^n}{3^{n-1-1}(b+c+c+a+a+b)} \\ &= \left(\frac{2}{3} \right)^{n-2} S^{n-1} \end{aligned}$$

Fractional Integrals and Fractional Derivatives

Let $0 < m < n \leq \infty$. We denote the space of all functions on $[m, n]$ having continuous derivatives up to m order by $C^m([m, n])$, where $AC([m, n])$ is the space of all absolutely continuous function on the interval $[m, n]$. $AC^m([m, n])$ is the space of all functions $g \in C^{l-1}([m, n])$ where $g^{(l-1)} \in AC([m, n])$. For $\alpha \in \mathbb{R}$, $[\alpha]$ is the integral part of α (the integer part of k such that $k \leq \alpha < k+1$). Also, $\lceil \alpha \rceil$ is the ceiling of α ($\min j \in \mathbb{N}, N \geq \alpha$).

We denote the space of all integrable functions on the interval (m, n) by $L_1(m, n)$ and the set of all functions that are measurable and essentially bounded on the interval (m, n) is denoted by $L_\infty(m, n)$. Evidently, $L_\infty(m, n) \subset L_1(m, n)$.

The following definition is given in [10, p.110].^[15]

Definition 1.3.1.

Let $[m, n]$ be finite or infinite of \square_+ and $\alpha > 0$. The left and right-sided Hadamard-type fractional integrals of order $\alpha > 0$ are given by

$$J_{m+}^\alpha y(s) = \frac{1}{\Gamma} \int_m^s \left(\log \frac{s}{t} \right)^{\alpha-1} \frac{y(t) dt}{t}, \quad s > m \quad (1.3.1)$$

and

$$J_{n-}^{\alpha} y(s) = \frac{1}{\Gamma_s} \int_s^n \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{y(t) dt}{t}, \quad s < n \quad (1.3.2)$$

respectively.

Definition 1.3.3

[5] Let $AC([a, b])$ be the space of all absolutely continuous functions on $[a, b]$. We denote the space of all functions $f \in C^n([a, b])$ with $f^{(n-1)} \in AC([a, b])$ by $AC^n([a, b])$. Let $\alpha \in \mathbb{R}^+$ & $f \in AC^n([a, b])$, then the Caputo fractional derivative of order α for a function $f(t)$ is defined as

$$D_{*a}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^{(n)}(x)}{(t-x)^{\alpha-n+1}} dx \quad (1.3.3)$$

where, $n = [\alpha] + 1$; the notation of $[\alpha]$ stands for the largest integer not greater than $[\alpha]$.

Lemma 1.3.4

[5] Let $\beta > \alpha \geq 0$. $f \in L_1(m, n)$ has an L_{∞} fractional derivative D_{α}^{β} in the interval $[m, n]$ & $D_m^{\beta-k} f(m) = 0$, $k=1, \dots, [\beta]+1$.

Then,

$$D_m^{\alpha} f(x) = \frac{1}{\Gamma(\beta-\alpha)} \int_m^x (x-y)^{\beta-\alpha-1} D_{\alpha}^{\beta} f(y) dy$$

$$\forall m \leq x \leq n$$

Clearly,

$$D_m^{\alpha} f \text{ is in } AC([m, n]) \text{ for } \beta - \alpha \geq 1,$$

$$D_m^{\alpha} f \text{ is in } C([m, n]) \text{ for } \beta - \alpha \in (0, 1).$$

Hence,

$$D_m^{\alpha} f \in L_{\infty}(m, n),$$

$$D_m^{\alpha} f \in L_1(m, n).$$

Mean Value Theorem for Fractional Integrals

Here we give the Mean Value Theorem for Riemann-Liouville Fractional Integrals and Hadamard-type Fractional Integrals.

2.1.1. Mean Value Theorem for Riemann-Liouville Fractional Integrals

Theorem 2.1.2.

Let $f, g \in C^2(I)$ where I is a compact interval, $u_i \in C([a, b]) (i = 1, 2)$, and $r(x) \geq 0 \forall x \in [a, b]$. Also let, $\frac{I_a^\alpha u(x)}{I_a^\alpha u(x)}, \frac{u(x)}{u(x)} \in I$, let $\frac{u(x)}{u(x)}$ be a non-constant. Let $Q_I(t)$ be given in (2.1.2), and $u_1(x), u_2(x)$ have Riemann-Liouville fractional integral of order $\alpha > 0$.

Then $\exists \zeta \in I$ such that

$$\frac{\int_a^b Q_I(x) f\left(\frac{u(x)}{u(x)}\right) dx - \int_a^b r(x) f\left(\frac{I_a^\alpha u(x)}{I_a^\alpha u(x)}\right) dx}{\int_a^b Q_I(x) g\left(\frac{u(x)}{u(x)}\right) dx - \int_a^b r(x) g\left(\frac{I_a^\alpha u(x)}{I_a^\alpha u(x)}\right) dx} = \frac{f''(\zeta)}{g''(\zeta)} \quad (2.1.2)$$

Provided that the denominators are not equal to zero.

Proof. Let us take a function $h \in C^2(I)$ defined as

$$h(x) = c_1 f(x) - c_2 g(x) \quad (2.1.3)$$

where

$$c_1 = \int_a^b Q_I(x) g\left(\frac{u_1(x)}{u_2(x)}\right) dx - \int_a^b r(x) g\left(\frac{I_a^\alpha u_1(x)}{I_a^\alpha u_2(x)}\right) dx$$

$$c_2 = \int_a^b Q_I(x) f\left(\frac{u_1(x)}{u_2(x)}\right) dx - \int_a^b r(x) f\left(\frac{I_a^\alpha u_1(x)}{I_a^\alpha u_2(x)}\right) dx$$

By Theorem 3.1.1 with $h = f$, we have

$$c_1 \int_a^b Q_I(x) f\left(\frac{u_1(x)}{u_2(x)}\right) dx - c_1 \int_a^b r(x) f\left(\frac{I_a^\alpha u_2(x)}{I_a^\alpha u_2(x)}\right) dx - c_2 \int_a^b Q_I(x) g\left(\frac{u_1(x)}{u_2(x)}\right) dx + c_2 \int_a^b r(x) g\left(\frac{I_a^\alpha u_2(x)}{I_a^\alpha u_2(x)}\right) dx$$

$$= \frac{c_1 f''(\zeta)}{2} \int_a^b Q_I(x) \left(\frac{u_1(x)}{u_2(x)}\right)^2 dx - \frac{c_1 f''(\zeta)}{2} \int_a^b r(x) f\left(\frac{I_a^\alpha u_2(x)}{I_a^\alpha u_2(x)}\right)^2 dx - \frac{c_2 g''(\zeta)}{2} \int_a^b Q_I(x) \left(\frac{u_1(x)}{u_2(x)}\right)^2 dx + \frac{c_2 g''(\zeta)}{2} \int_a^b r(x) f\left(\frac{I_a^\alpha u_2(x)}{I_a^\alpha u_2(x)}\right)^2 dx$$

Putting the values of c_1 and c_2 in above equation, we get

$$\Rightarrow 0 = \frac{c_1 f''(\zeta)}{2} \int_a^b Q_I(x) \left(\frac{u_1(x)}{u_2(x)}\right)^2 dx - \frac{c_1 f''(\zeta)}{2} \int_a^b r(x) \left(\frac{I_a^\alpha u_2(x)}{I_a^\alpha u_2(x)}\right)^2 dx - \frac{c_2 g''(\zeta)}{2} \int_a^b Q_I(x) \left(\frac{u_1(x)}{u_2(x)}\right)^2 dx + \frac{c_2 g''(\zeta)}{2} \int_a^b r(x) \left(\frac{I_a^\alpha u_2(x)}{I_a^\alpha u_2(x)}\right)^2 dx$$

$$\Rightarrow 0 = \left(\frac{c_1 f''(\zeta)}{2} - \frac{c_2 g''(\zeta)}{2}\right) \left(\int_a^b Q_I(x) \left(\frac{u_1(x)}{u_2(x)}\right)^2 dx - \int_a^b r(x) \left(\frac{I_a^\alpha u_2(x)}{I_a^\alpha u_2(x)}\right)^2 dx\right)$$

since

$$\int_a^b Q_I(x) \left(\frac{u_1(x)}{u_2(x)}\right)^2 dx - \int_a^b r(x) f\left(\frac{I_a^\alpha u_2(x)}{I_a^\alpha u_2(x)}\right)^2 dx \neq 0,$$

so we get,

$$\begin{aligned}
& \frac{c_1 f''(\zeta)}{2} - \frac{c_2 g''(\zeta)}{2} = 0 \\
& \Rightarrow c_1 f''(\zeta) - c_2 g''(\zeta) = 0 \\
& \Rightarrow \frac{c_2}{c_1} = \frac{f''(\zeta)}{g''(\zeta)} \\
& \Rightarrow \frac{\int_a^b Q_I(x) f\left(\frac{u(x)}{u(x)}\right) dx - \int_a^b r(x) f\left(\frac{I_a^\alpha u(x)}{I_a^\alpha u(x)}\right) dx}{\int_a^b Q_I(x) g\left(\frac{u(x)}{u(x)}\right) dx - \int_a^b r(x) g\left(\frac{I_a^\alpha u(x)}{I_a^\alpha u(x)}\right) dx} = \frac{f''(\zeta)}{g''(\zeta)} \quad (2.1.4)
\end{aligned}$$

This completes the proof.

Theorem 3.1.1.

Let $I \subseteq \mathbb{R}^+$ where I is a compact interval, $r(x) \geq 0 \forall x \in [a, b]$ and

$\frac{I_a^\alpha v(x)}{I_a^\alpha v(x)}, \frac{v(x)}{v(x)} \in I$ where $\frac{v(x)}{v(x)}$ is non-constant and let $Q_I(t)$ be given in (2.1.2).

Then $\forall p, q \in \mathbb{R} \setminus \{0, 1\}$ and $p \neq q$, $\exists \zeta$ such that

$$\left(\frac{p(p-1) \int_a^b Q_I(x) \left(\frac{v(x)}{v(x)}\right)^q dx - \int_a^b r(x) \left(\frac{I_a^\alpha v(x)}{I_a^\alpha v(x)}\right)^q dx}{q(q-1) \int_a^b Q_I(x) \left(\frac{v(x)}{v(x)}\right)^p dx - \int_a^b r(x) \left(\frac{I_a^\alpha v(x)}{I_a^\alpha v(x)}\right)^p dx} \right)^{q-p} \quad (3.1.1)$$

Proof.

We set, $f(x) = x^q$ and $g(x) = x^p$, $q \neq p$, $p, q \neq 0, 1$. By the Theorem (3.1.2) we have

$$\begin{aligned}
& \frac{\int_a^b Q_I(x) \left(\frac{v(x)}{v(x)}\right)^q dx - \int_a^b r(x) \left(\frac{I_a^\alpha v(x)}{I_a^\alpha v(x)}\right)^q dx}{\int_a^b Q_I(x) \left(\frac{v(x)}{v(x)}\right)^p dx - \int_a^b r(x) \left(\frac{I_a^\alpha v(x)}{I_a^\alpha v(x)}\right)^p dx} = \frac{q(q-1)\zeta^{q-2}}{p(p-1)\zeta^{p-2}} \\
& \Rightarrow \zeta^{q-p} = \frac{p(p-1) \int_a^b Q_I(x) \left(\frac{v(x)}{v(x)}\right)^q dx - \int_a^b r(x) \left(\frac{I_a^\alpha v(x)}{I_a^\alpha v(x)}\right)^q dx}{q(q-1) \int_a^b Q_I(x) \left(\frac{v(x)}{v(x)}\right)^p dx - \int_a^b r(x) \left(\frac{I_a^\alpha v(x)}{I_a^\alpha v(x)}\right)^p dx}
\end{aligned}$$

Or,

$$\Rightarrow \zeta = \left(\frac{p(p-1) \int_a^b Q_l(x) \left(\frac{v(x)}{v(x)}\right)^q dx - \int_a^b r(x) \left(\frac{I_a^\alpha v(x)}{I_a^\alpha v(x)}\right)^q dx}{q(q-1) \int_a^b Q_l(x) \left(\frac{v(x)}{v(x)}\right)^p dx - \int_a^b r(x) \left(\frac{I_a^\alpha v(x)}{I_a^\alpha v(x)}\right)^p dx} \right)^{\overline{q-p}}$$

Hence the proof is complete.

3.1.2. Mean Value Theorem for the Hadamard-type Fractional Integrals

Theorem 3.1.3.

Let, $f, g \in C^2(I)$, and I be a compact interval, $f_i \in C([a, b]) (i=1, 2)$ and

$r(s) \geq 0 \forall s \in [a, b]$. Also let $\frac{f(t)}{f(t)}, \frac{J_a^\delta f(s)}{J_a^\delta f(s)} \in I$, let $\frac{f(t)}{f(t)}$ be nonconstant, let $Z(t)$ be given

in (2.1.2), and $f_1(t), f_2(t)$ have Hadamard-type fractional integral of order $\alpha > 0$. Then there exist $\exists \zeta \in I$ such that

$$\frac{\int_a^b Z(t) f \left\{ \frac{f(t)}{f(t)} \right\} ds - \int_a^b r(s) f \left\{ \frac{J_a^\delta f(s)}{J_a^\delta f(s)} \right\} ds}{\int_a^b Z(t) g \left\{ \frac{f(t)}{f(t)} \right\} ds - \int_a^b r(s) g \left\{ \frac{J_a^\delta f(s)}{J_a^\delta f(s)} \right\} ds} = \frac{f''(\zeta)}{g''(\zeta)} \quad (3.1.3)$$

provided that denominators are not equal to zero.

Proof.

Let us take a function $h \in C^2(I)$ defined as

$$h(s) = c_1 f(s) - c_2 g(s) \quad (3.1.4)$$

where

$$c_1 = \int_a^b Z(t) g \left\{ \frac{f_1(t)}{f_2(t)} \right\} ds - \int_a^b r(s) g \left\{ \frac{J_{a+}^\delta f_1(s)}{J_{a+}^\delta f_2(s)} \right\} ds,$$

$$c_2 = \int_a^b Z(t) f \left\{ \frac{f_1(t)}{f_2(t)} \right\} ds - \int_a^b r(s) f \left\{ \frac{J_{a+}^\delta f_1(s)}{J_{a+}^\delta f_2(s)} \right\} ds,$$

By Theorem 3.1.4 with $f = h$, we get

$$c_1 \int_a^b Z(t) f \left\{ \frac{f_1(t)}{f_2(t)} \right\} ds - c_1 \int_a^b r(s) f \left\{ \frac{J_{a+}^\delta f_1(s)}{J_{a+}^\delta f_2(s)} \right\} ds - c_2 \int_a^b Z(t) g \left\{ \frac{f_1(t)}{f_2(t)} \right\} ds + c_2 \int_a^b r(s) g \left\{ \frac{J_{a+}^\delta f_1(s)}{J_{a+}^\delta f_2(s)} \right\} ds$$

$$= \frac{c_1 f''(\zeta)}{2} \int_a^b Z(t) \left\{ \frac{f_1(t)}{f_2(t)} \right\}^2 ds - \frac{c_1 f''(\zeta)}{2} \int_a^b r(s) \left\{ \frac{J_{a+}^\delta f_1(s)}{J_{a+}^\delta f_2(s)} \right\}^2 ds - \frac{c_2 g''(\zeta)}{2} \int_a^b Z(t) \left\{ \frac{f_1(t)}{f_2(t)} \right\}^2 ds + \frac{c_2 g''(\zeta)}{2} \int_a^b r(s) \left\{ \frac{J_{a+}^\delta f_1(s)}{J_{a+}^\delta f_2(s)} \right\}^2 ds$$

Putting the values of C_1 and C_2 in above equation, we get

$$0 = \left\{ \frac{c_1 f''(\zeta)}{2} - \frac{c_2 g''(\zeta)}{2} \right\} \left[\int_a^b Z(t) \left\{ \frac{f_1(t)}{f_2(t)} \right\}^2 ds - \int_a^b r(s) \left\{ \frac{J_{a+}^\delta f_1(s)}{J_{a+}^\delta f_2(s)} \right\}^2 ds \right].$$

Since

$$\int_a^b Z(t) \left\{ \frac{f_1(t)}{f_2(t)} \right\}^2 ds - \int_a^b r(s) \left\{ \frac{J_{a+}^\delta f_1(s)}{J_{a+}^\delta f_2(s)} \right\}^2 ds \neq 0,$$

So, we have

$$\frac{c_1 f''(\zeta)}{2} - \frac{c_2 g''(\zeta)}{2} = 0$$

$$c_1 f''(\zeta) - c_2 g''(\zeta) = 0$$

$$\frac{c_2}{c_1} = \frac{f''(\zeta)}{g''(\zeta)}$$

$$\Rightarrow \frac{\int_a^b Z(t) f \left\{ \frac{f(t)}{f(t)} \right\} ds - f \int_a^b r(s) \left\{ \frac{J_a^\delta f(s)}{J_a^\delta f(s)} \right\} ds}{\int_a^b Z(t) g \left\{ \frac{f(t)}{f(t)} \right\} ds - g \int_a^b r(s) \left\{ \frac{J_a^\delta f(s)}{J_a^\delta f(s)} \right\} ds} = \frac{f''(\zeta)}{g''(\zeta)} \quad (3.1.5)$$

Hence the proof is complete.

Theorem 3.1.6

Let $I \subseteq \mathbb{R}^+$ and I be the compact interval where $r(s) \geq 0 \forall s \in [a, b]$. Let

$\frac{f(t)}{f(t)}, \frac{J_a^\delta f(s)}{J_a^\delta f(s)} \in I$ and $\frac{f(t)}{f(t)}$ be the non-constant. And let $Z(y)$ be given in(2.1.4). Then for

$p, q \in \mathbb{R} \setminus \{0, 1\}$ and $p \neq q, \exists \zeta$ such that

$$\zeta = \left[\frac{p(p-1) \int_a^b Z(t) \left\{ \frac{f(t)}{f(t)} \right\}^q ds - \int_a^b r(s) \left\{ \frac{J_{a+}^\delta f(s)}{J_{a+}^\delta f(s)} \right\}^q ds}{q(q-1) \int_a^b Z(t) \left\{ \frac{f(t)}{f(t)} \right\}^p ds - \int_a^b r(s) \left\{ \frac{J_{a+}^\delta f(s)}{J_{a+}^\delta f(s)} \right\}^p ds} \right]^{q-p} \quad (3.1.7)$$

Proof.

We set, $f(s) = s^q$ and $g(s) = s^p$ where, $p \neq q, p, q \neq 0, 1$. By the theorem (3.1.5) we have

$$\left[\frac{\int_a^b Z(t) \left\{ \frac{f(t)}{f(t)} \right\}^q ds - \int_a^b r(s) \left\{ \frac{J_{a+}^\delta f(s)}{J_{a+}^\delta f(s)} \right\}^q ds}{\int_a^b Z(t) \left\{ \frac{f(t)}{f(t)} \right\}^p ds - \int_a^b r(s) \left\{ \frac{J_{a+}^\delta f(s)}{J_{a+}^\delta f(s)} \right\}^p ds} \right] = \frac{q(q-1)\zeta^{q-2}}{p(p-1)\zeta^{p-2}}$$

$$\Rightarrow \zeta^{q-p} = \left[\frac{p(p-1) \int_a^b Z(t) \left\{ \frac{f(t)}{f(t)} \right\}^q ds - \int_a^b r(s) \left\{ \frac{J_{a+}^\delta f(s)}{J_{a+}^\delta f(s)} \right\}^q ds}{q(q-1) \int_a^b Z(t) \left\{ \frac{f(t)}{f(t)} \right\}^p ds - \int_a^b r(s) \left\{ \frac{J_{a+}^\delta f(s)}{J_{a+}^\delta f(s)} \right\}^p ds} \right]$$

Or,

$$\zeta = \left[\frac{p(p-1) \int_a^b Z(t) \left\{ \frac{f(t)}{f(t)} \right\}^q ds - \int_a^b r(s) \left\{ \frac{J_{a+}^\delta f(s)}{J_{a+}^\delta f(s)} \right\}^q ds}{q(q-1) \int_a^b Z(t) \left\{ \frac{f(t)}{f(t)} \right\}^p ds - \int_a^b r(s) \left\{ \frac{J_{a+}^\delta f(s)}{J_{a+}^\delta f(s)} \right\}^p ds} \right]^{q-p}$$

This completes the proof.

3.2. Mean Value Theorems for Differential Fractional Derivatives

We give the Mean Value Theorems for the Caputo Fractional Derivatives and L_∞ Caputo Fractional Derivatives in this section.

3.2.1 Mean Value Theorems for the Caputo Fractional Derivatives

Theorem 3.2.2

Let $f, g \in C^2(I)$, and I be a compact interval, $w_i \in AC^n([a, b]) (i=1, 2)$, and

$r(x) \geq 0 \forall x \in [a, b]$. Also let $\frac{w^{(n)}(x)}{w^{(n)}(x)}, \frac{D_{*a}^\alpha w(x)}{D_{*a}^\alpha w(x)} \in I$, let $\frac{w^{(n)}(x)}{w^{(n)}(x)}$ be the non-constant, let

$Q_D(t)$, be given in (2.2.2) and $w_1(x), w_2(x)$ have the Caputo derivative of order $\alpha > 0$. Then

$\exists \zeta \in I$ such that

$$\frac{\int_a^b Q_D(x) f \left\{ \frac{w^{(n)}(x)}{w^{(n)}(x)} \right\} dx - \int_a^b r(x) f \left\{ \frac{D_{*a}^\alpha w(x)}{D_{*a}^\alpha w(x)} \right\} dx}{\int_a^b Q_D(x) g \left\{ \frac{w^{(n)}(x)}{w^{(n)}(x)} \right\} dx - \int_a^b r(x) g \left\{ \frac{D_{*a}^\alpha w(x)}{D_{*a}^\alpha w(x)} \right\} dx} = \frac{f''(\zeta)}{g''(\zeta)} \quad (3.2.2)$$

provided that denominators are not equal to zero.

Proof.

Let us take a function $h \in C^2(I)$ defined as

$$h(x) = c_1 f(x) - c_2 g(x) \quad (3.2.3)$$

where,

$$c_1 = \int_a^b Q_D(x) g \left\{ \frac{w_1^{(n)}(x)}{w_2^{(n)}(x)} \right\} dx - \int_a^b r(x) g \left\{ \frac{D_{*a}^\alpha w_1(x)}{D_{*a}^\alpha w_2(x)} \right\} dx$$

$$c_2 = \int_a^b Q_D(x) f \left\{ \frac{w_1^{(n)}(x)}{w_2^{(n)}(x)} \right\} dx - \int_a^b r(x) f \left\{ \frac{D_{*a}^\alpha w_1(x)}{D_{*a}^\alpha w_2(x)} \right\} dx$$

By Theorem (3.2.1) with $f = h$, we have

$$c_1 \int_a^b Q_D(x) f \left\{ \frac{w_1^{(n)}(x)}{w_2^{(n)}(x)} \right\} dx - c_1 \int_a^b r(x) f \left\{ \frac{D_{*a}^\alpha w_1(x)}{D_{*a}^\alpha w_2(x)} \right\} dx - c_2 \int_a^b Q_D(x) g \left\{ \frac{w_1^{(n)}(x)}{w_2^{(n)}(x)} \right\} dx + c_2 \int_a^b r(x) g \left\{ \frac{D_{*a}^\alpha w_1(x)}{D_{*a}^\alpha w_2(x)} \right\} dx$$

$$= \frac{c_1 f''(\zeta)}{2} \int_a^b Q_D(x) \left\{ \frac{w_1^{(n)}(x)}{w_2^{(n)}(x)} \right\}^2 dx - \frac{c_1 f''(\zeta)}{2} \int_a^b r(x) \left\{ \frac{D_{*a}^\alpha w_1(x)}{D_{*a}^\alpha w_2(x)} \right\}^2 dx - \frac{c_2 g''(\zeta)}{2} \int_a^b Q_D(x) \left\{ \frac{w_1^{(n)}(x)}{w_2^{(n)}(x)} \right\}^2 dx + \frac{c_2 g''(\zeta)}{2} \int_a^b r(x) \left\{ \frac{D_{*a}^\alpha w_1(x)}{D_{*a}^\alpha w_2(x)} \right\}^2 dx$$

Putting the values of c_1 and c_2 in the above equation, we now have

$$0 = \left\{ \frac{c_1 f''(\zeta)}{2} - \frac{c_2 g''(\zeta)}{2} \right\} \left[\int_a^b Q_D(x) \left\{ \frac{w_1^{(n)}(x)}{w_2^{(n)}(x)} \right\}^2 dx - \int_a^b r(x) \left\{ \frac{D_{*a}^\alpha w_1(x)}{D_{*a}^\alpha w_2(x)} \right\}^2 dx \right]$$

Since,

$$\int_a^b Q_D(x) \left\{ \frac{w_1^{(n)}(x)}{w_2^{(n)}(x)} \right\}^2 dx - \int_a^b r(x) \left\{ \frac{D_{*a}^\alpha w_1(x)}{D_{*a}^\alpha w_2(x)} \right\}^2 dx \neq 0,$$

So, we have

$$\frac{c_1 f''(\zeta)}{2} - \frac{c_2 g''(\zeta)}{2} = 0$$

$$c_1 f''(\zeta) - c_2 g''(\zeta) = 0$$

$$\frac{c_2}{c_1} = \frac{f''(\zeta)}{g''(\zeta)}$$

$$\Rightarrow \frac{\int_a^b Q_D(x) f \left\{ \frac{w_1^{(n)}(x)}{w_2^{(n)}(x)} \right\} dx - \int_a^b r(x) f \left\{ \frac{D_{*a}^\alpha w_1(x)}{D_{*a}^\alpha w_2(x)} \right\} dx}{\int_a^b Q_D(x) g \left\{ \frac{w_1^{(n)}(x)}{w_2^{(n)}(x)} \right\} dx - \int_a^b r(x) g \left\{ \frac{D_{*a}^\alpha w_1(x)}{D_{*a}^\alpha w_2(x)} \right\} dx} = \frac{f''(\zeta)}{g''(\zeta)} \quad (3.2.4)$$

This completes the proof.

Theorem 3.2.5

Let $I \subseteq \mathbb{R}^+$ and I be the compact interval, $r(x) \geq 0 \forall x \in [a, b]$. Let $\frac{D_{*a}^\alpha w(x)}{D_{*a}^\alpha w(x)}, \frac{w^{(n)}(x)}{w^{(n)}(x)} \in I$

and $\frac{w^{(n)}(x)}{w^{(n)}(x)}$ be the non-constant. Also let $Q_D(t)$ be given in (2.2.2). Then,

for $p, q \in \mathbb{R} \setminus \{0, 1\}$ and $p \neq q$, $\exists \zeta \in I$ such that

$$\zeta = \frac{\left[p(p-1) \int_a^b Q_D(x) \left\{ \frac{w^{(n)}(x)}{w^{(n)}(x)} \right\}^q dx - \int_a^b r(x) \left\{ \frac{D_{*a}^\alpha w(x)}{D_{*a}^\alpha w(x)} \right\}^q dx \right]^{q-p}}{\left[q(q-1) \int_a^b Q_D(x) \left\{ \frac{w^{(n)}(x)}{w^{(n)}(x)} \right\}^p dx - \int_a^b r(x) \left\{ \frac{D_{*a}^\alpha w(x)}{D_{*a}^\alpha w(x)} \right\}^p dx \right]} \quad (3.2.5)$$

Proof.

We set $f(x) = x^q$ and $g(x) = x^p$, $p \neq q$, $p, q \neq 0, 1$. By theorem (3.2.2) we have

$$\frac{\int_a^b Q_D(x) \left\{ \frac{w^{(n)}(x)}{w^{(n)}(x)} \right\}^q dx - \int_a^b r(x) \left\{ \frac{D_{*a}^\alpha w(x)}{D_{*a}^\alpha w(x)} \right\}^q dx}{\int_a^b Q_D(x) \left\{ \frac{w^{(n)}(x)}{w^{(n)}(x)} \right\}^p dx - \int_a^b r(x) \left\{ \frac{D_{*a}^\alpha w(x)}{D_{*a}^\alpha w(x)} \right\}^p dx} = \frac{q(q-1)\zeta^{q-2}}{p(p-1)\zeta^{p-2}},$$

$$\Rightarrow \zeta^{q-p} = \frac{p(p-1) \int_a^b Q_D(x) \left\{ \frac{w^{(n)}(x)}{w^{(n)}(x)} \right\}^q dx - \int_a^b r(x) \left\{ \frac{D_{*a}^\alpha w(x)}{D_{*a}^\alpha w(x)} \right\}^q dx}{q(q-1) \int_a^b Q_D(x) \left\{ \frac{w^{(n)}(x)}{w^{(n)}(x)} \right\}^p dx - \int_a^b r(x) \left\{ \frac{D_{*a}^\alpha w(x)}{D_{*a}^\alpha w(x)} \right\}^p dx},$$

or

$$\zeta = \frac{\left[p(p-1) \int_a^b Q_D(x) \left\{ \frac{w^{(n)}(x)}{w^{(n)}(x)} \right\}^q dx - \int_a^b r(x) \left\{ \frac{D_{*a}^\alpha w(x)}{D_{*a}^\alpha w(x)} \right\}^q dx \right]^{q-p}}{\left[q(q-1) \int_a^b Q_D(x) \left\{ \frac{w^{(n)}(x)}{w^{(n)}(x)} \right\}^p dx - \int_a^b r(x) \left\{ \frac{D_{*a}^\alpha w(x)}{D_{*a}^\alpha w(x)} \right\}^p dx \right]}.$$

Hence the proof is complete.

Mean Value Theorems for L_∞ Fractional Derivative**Theorem 3.2.6**

Let $\beta > \alpha \geq 0$, $f, g \in C^2(I)$ let I be a compact interval $u_i \in L_1(a, b) (i=1, 2)$ has an L_∞

fractional derivative and $r(x) \geq 0 \forall x \in [a, b]$. Let $D_a^{\beta-k} w_i(a) = 0$ for

$k = 1, \dots, [\beta] + 1 (i = 1, 2)$, $\frac{D_a^\beta w(x)}{D_a^\beta w(x)}, \frac{D_a^\alpha w(x)}{D_a^\alpha w(x)} \in I$, let $\frac{D_a^\beta w(x)}{D_a^\beta w(x)}$ be the non-constant, and let

$Q_L(t)$ be given in (2.2.5). Then $\exists \zeta \in I$ such that

$$\frac{\int_a^b \left[Q_L(x) f \left\{ \frac{D_a^\beta w(x)}{D_a^\beta w(x)} \right\} \right] dx - \int_a^b r(x) f \left\{ \frac{D_a^\alpha w(x)}{D_a^\alpha w(x)} \right\} dx}{\int_a^b \left[Q_L(x) g \left\{ \frac{D_a^\beta w(x)}{D_a^\beta w(x)} \right\} \right] dx - \int_a^b r(x) g \left\{ \frac{D_a^\alpha w(x)}{D_a^\alpha w(x)} \right\} dx} = \frac{f''(\zeta)}{g''(\zeta)} \quad (3.2.6).$$

provided that denominators are not equal to zero.

Proof.

Let us take a function $h \in C^2(I)$ defined as

$$h(x) = c_1 f(x) - c_2 g(x) \quad (3.2.7)$$

where

$$c_1 = \int_a^b \left[Q_L(x) g \left\{ \frac{D_a^\beta w_1(x)}{D_a^\beta w_2(x)} \right\} \right] dx - \int_a^b r(x) g \left\{ \frac{D_a^\alpha w_1(x)}{D_a^\alpha w_2(x)} \right\} dx$$

$$c_2 = \int_a^b \left[Q_L(x) f \left\{ \frac{D_a^\beta w_1(x)}{D_a^\beta w_2(x)} \right\} \right] dx - \int_a^b r(x) f \left\{ \frac{D_a^\alpha w_1(x)}{D_a^\alpha w_2(x)} \right\} dx$$

By Theorem 3.2.4 with $f = h$, we have

$$\begin{aligned} & c_1 \int_a^b \left[Q_L(x) f \left\{ \frac{D_a^\beta w_1(x)}{D_a^\beta w_2(x)} \right\} \right] dx - c_1 \int_a^b r(x) f \left\{ \frac{D_a^\alpha w_1(x)}{D_a^\alpha w_2(x)} \right\} dx \\ & - c_2 \int_a^b \left[Q_L(x) g \left\{ \frac{D_a^\beta w_1(x)}{D_a^\beta w_2(x)} \right\} \right] dx + c_2 \int_a^b r(x) g \left\{ \frac{D_a^\alpha w_1(x)}{D_a^\alpha w_2(x)} \right\} dx \\ & = \frac{c_1 f''(\zeta)}{2} \int_a^b \left[Q_L(x) \left\{ \frac{D_a^\beta w_1(x)}{D_a^\beta w_2(x)} \right\}^2 \right] dx - \frac{c_1 f''(\zeta)}{2} \int_a^b r(x) \left\{ \frac{D_a^\alpha w_1(x)}{D_a^\alpha w_2(x)} \right\}^2 dx \\ & - \frac{c_2 g''(\zeta)}{2} \int_a^b \left[Q_L(x) \left\{ \frac{D_a^\beta w_1(x)}{D_a^\beta w_2(x)} \right\}^2 \right] dx + \frac{c_2 g''(\zeta)}{2} \int_a^b r(x) \left\{ \frac{D_a^\alpha w_1(x)}{D_a^\alpha w_2(x)} \right\}^2 dx \end{aligned}$$

Putting the values of c_1 and c_2 in above equation, we get

$$0 = \left\{ \frac{c_1 f''(\zeta)}{2} - \frac{c_2 g''(\zeta)}{2} \right\} \left[\int_a^b Q_L(x) \left\{ \frac{D_a^\beta w_1(x)}{D_a^\beta w_2(x)} \right\}^2 dx - \int_a^b r(x) \left\{ \frac{D_a^\alpha w_1(x)}{D_a^\alpha w_2(x)} \right\}^2 dx \right]$$

Since,

$$\int_a^b Q_L(x) \left\{ \frac{D_a^\beta w_1(x)}{D_a^\beta w_2(x)} \right\}^2 dx - \int_a^b r(x) \left\{ \frac{D_a^\alpha w_1(x)}{D_a^\alpha w_2(x)} \right\}^2 dx \neq 0$$

So, we have

$$\frac{c_1 f''(\zeta)}{2} - \frac{c_2 g''(\zeta)}{2} = 0$$

$$c_1 f''(\zeta) - c_2 g''(\zeta) = 0$$

$$\frac{c_2}{c_1} = \frac{f''(\zeta)}{g''(\zeta)}$$

$$\Rightarrow \frac{\int_a^b Q_L(x) f \left\{ \frac{D_a^\beta w(x)}{D_a^\beta w(x)} \right\} dx - \int_a^b r(x) f \left\{ \frac{D_a^\alpha w(x)}{D_a^\alpha w(x)} \right\} dx}{\int_a^b Q_L(x) g \left\{ \frac{D_a^\beta w(x)}{D_a^\beta w(x)} \right\} dx - \int_a^b r(x) g \left\{ \frac{D_a^\alpha w(x)}{D_a^\alpha w(x)} \right\} dx} = \frac{f''(\zeta)}{g''(\zeta)} \quad (3.2.24)$$

This completes the proof.

Theorem 3.2.8.

Let, $I \subseteq \mathbb{R}^+$, and I be a compact interval, $u_i \in U(v, k) (i = 1, 2)$, and

$r(x) \geq 0 \forall x \in [a, b]$. Let, $\frac{D_a^\beta w(x)}{D_a^\beta w(x)}, \frac{D_a^\alpha w(x)}{D_a^\alpha w(x)} \in I$, $\frac{D_a^\beta w(x)}{D_a^\beta w(x)}$ be non-constant and let that

$Q_L(t)$ be given in (2.2.5). Then for $p, q \in \mathbb{R} \setminus \{0, 1\}$ and $p \neq q$, $\exists \zeta \in I$ such that,

$$\zeta = \frac{\left[p(p-1) \int_a^b Q_L(x) \left\{ \frac{D_a^\beta w(x)}{D_a^\beta w(x)} \right\}^q dx - \int_a^b r(x) \left\{ \frac{D_a^\alpha w(x)}{D_a^\alpha w(x)} \right\}^q dx \right]^{q-p}}{\left[q(q-1) \int_a^b Q_L(x) \left\{ \frac{D_a^\beta w(x)}{D_a^\beta w(x)} \right\}^p dx - \int_a^b r(x) \left\{ \frac{D_a^\alpha w(x)}{D_a^\alpha w(x)} \right\}^p dx \right]} \quad (3.2.9)$$

Proof.

We set $f(x) = x^q$ and $g(x) = x^p$, $p \neq q$, $p, q \neq 0, 1$. By theorem 3.2.5 we have

$$\frac{\int_a^b Q_L(x) \left\{ \frac{D_a^\beta w(x)}{D_a^\beta w(x)} \right\}^q dx - \int_a^b r(x) \left\{ \frac{D_a^\alpha w(x)}{D_a^\alpha w(x)} \right\}^q dx}{\int_a^b Q_L(x) \left\{ \frac{D_a^\beta w(x)}{D_a^\beta w(x)} \right\}^p dx - \int_a^b r(x) \left\{ \frac{D_a^\alpha w(x)}{D_a^\alpha w(x)} \right\}^p dx} = \frac{q(q-1)\zeta^{q-2}}{p(p-1)\zeta^{p-2}} \quad (3.2.10)$$

$$\Rightarrow \zeta^{q-p} = \frac{p(p-1) \int_a^b \left[Q_L(x) \left\{ \frac{D_a^\beta w(x)}{D_a^\beta w(x)} \right\}^q \right] dx - \int_a^b r(x) \left\{ \frac{D_a^\alpha w(x)}{D_a^\alpha w(x)} \right\}^q dx}{q(q-1) \int_a^b \left[Q_L(x) \left\{ \frac{D_a^\beta w(x)}{D_a^\beta w(x)} \right\}^p \right] dx - \int_a^b r(x) \left\{ \frac{D_a^\alpha w(x)}{D_a^\alpha w(x)} \right\}^p dx} \tag{3.2.11}$$

Or,

$$\zeta = \left[\frac{p(p-1) \int_a^b \left[Q_L(x) \left\{ \frac{D_a^\beta w(x)}{D_a^\beta w(x)} \right\}^q \right] dx - \int_a^b r(x) \left\{ \frac{D_a^\alpha w(x)}{D_a^\alpha w(x)} \right\}^q dx}{q(q-1) \int_a^b \left[Q_L(x) \left\{ \frac{D_a^\beta w(x)}{D_a^\beta w(x)} \right\}^p \right] dx - \int_a^b r(x) \left\{ \frac{D_a^\alpha w(x)}{D_a^\alpha w(x)} \right\}^p dx} \right]^{q-p} \tag{3.2.12}$$

Hence the proof is complete.

Means for the Fractional Integrals

In this chapter, we give the Means for Riemann-Liouville fractional integrals and Hadamard-type fractional integrals. See^[6] for detailed reference.

4.1.1 Means for the Hadamard-type Fractional Integrals

The means for the Hadamard-type fractional integral is defined as,

$$M_{p,q} = \left(\frac{\Pi_q}{\Pi_p} \right)^{\overline{q-p}} \tag{4.1.1}$$

where $p, q \in \mathbb{R}$,

$$\Pi_q = p(p-1) \int_a^b Z(y) \left\{ \frac{f_1(y)}{f_2(y)} \right\}^q dx - \int_a^b r(x) \left\{ \frac{J_{a+}^\delta f_1(x)}{J_{a+}^\delta f_2(x)} \right\}^q dx$$

and

$$\Pi_p = q(q-1) \int_a^b Z(y) \left\{ \frac{f_1(y)}{f_2(y)} \right\}^p dx - \int_a^b r(x) \left\{ \frac{J_{a+}^\delta f_1(x)}{J_{a+}^\delta f_2(x)} \right\}^p dx$$

In the following, we will discuss limiting cases of above mean for $p, q \neq 0, 1$

$$\lim_{q \rightarrow p} M_{p,q} = M_{p,p} = \exp \left[\frac{\int_a^b Z(y) \left\{ \frac{f(y)^p}{f(y)^p} \right\} \log \left\{ \frac{f(y)}{f(y)} \right\} dx - \int_a^b r(x) \left\{ \frac{J_a^\delta f(x)^p}{J_a^\delta f(x)^p} \right\} \log \left\{ \frac{J_a^\delta f(x)}{J_a^\delta f(x)} \right\} dx}{\int_a^b Z(y) \left\{ \frac{f(y)^p}{f(y)^p} \right\} dx - \int_a^b r(x) \left\{ \frac{J_a^\delta f(x)^p}{J_a^\delta f(x)^p} \right\} dx} - \frac{2p-1}{p(p-1)} \right]$$

$$\lim_{p \rightarrow 0} M_{p,p} = M_{0,0}$$

$$= \exp \left[\frac{\int_a^b Z(y) \log^2 \left\{ \frac{f(y)}{f(y)} \right\} dx - \int_a^b r(x) \log^2 \left\{ \frac{J_a^\delta f(x)}{J_a^\delta f(x)} \right\} dx}{2 \int_a^b Z(y) \log \left\{ \frac{f(y)}{f(y)} \right\} dx - \int_a^b r(x) \log \left\{ \frac{J_a^\delta f(x)}{J_a^\delta f(x)} \right\} dx} + 1 \right]$$

$$\lim_{p \rightarrow 1} M_{p,p} = M_{1,1}$$

$$= \exp \left[\frac{\int_a^b Z(y) \left\{ \frac{f(y)}{f(y)} \right\} \log^2 \left\{ \frac{f(y)}{f(y)} \right\} dx - \int_a^b r(x) \left\{ \frac{J_a^\delta f(x)}{J_a^\delta f(x)} \right\} \log^2 \left\{ \frac{J_a^\delta f(x)}{J_a^\delta f(x)} \right\} dx}{2 \int_a^b Z(y) \left\{ \frac{f(y)}{f(y)} \right\} \log \left\{ \frac{f(y)}{f(y)} \right\} dx - \int_a^b r(x) \left\{ \frac{J_a^\delta f(x)}{J_a^\delta f(x)} \right\} \log \left\{ \frac{J_a^\delta f(x)}{J_a^\delta f(x)} \right\} dx} - 1 \right]$$

$$\lim_{q \rightarrow 0} M_{p,q} = M_{p,0}$$

$$= \exp \left[\frac{\int_a^b Z(y) \left\{ \frac{f(y)^p}{f(y)^p} \right\} dx - \int_a^b r(x) \left\{ \frac{J_a^\delta f(x)^p}{J_a^\delta f(x)^p} \right\} dx}{2 \int_a^b Z(y) \log \left\{ \frac{f(y)}{f(y)} \right\} dx - \int_a^b r(x) \log \left\{ \frac{J_a^\delta f(x)^p}{J_a^\delta f(x)^p} \right\} dx} p(p-1) \right]^{\bar{p}}$$

$$\lim_{q \rightarrow 1} M_{p,q} = M_{p,1}$$

$$= \exp \left[\frac{\int_a^b Z(y) \left\{ \frac{f(y)}{f(y)} \right\} \log \left\{ \frac{f(y)}{f(y)} \right\} dx - \int_a^b r(x) \left\{ \frac{J_a^\delta f(x)}{J_a^\delta f(x)} \right\} \log \left\{ \frac{J_a^\delta f(x)}{J_a^\delta f(x)} \right\} dx}{\int_a^b Z(y) \left\{ \frac{f(y)^p}{f(y)^p} \right\} dx - \int_a^b r(x) \left\{ \frac{J_a^\delta f(x)^p}{J_a^\delta f(x)^p} \right\} dx} p(p-1) \right]^{\bar{p}}$$

4.2 Means for the Fractional Derivatives

4.2.1 Means for Caputo fractional derivative

Right-hand side of (3.1.12) is the mean, then for the distinct $p, q \in \mathbb{R}$, it can be written as

$$M_{p,q} = \left(\frac{\Pi_q}{\Pi_p} \right)^{\bar{q-p}} \quad (4.2.1)$$

Where,

$$\Pi_q = p(p-1) \int_a^b Q_D(x) \left\{ \frac{w_1^{(n)}(x)}{w_2^{(n)}(x)} \right\}^q dx - \int_a^b r(x) \left\{ \frac{D_{*a}^\alpha w_1(x)}{D_{*a}^\alpha w_2(x)} \right\}^q dx$$

and

$$\Pi_p = q(q-1) \int_a^b Q_D(x) \left\{ \frac{w_1^{(n)}(x)}{w_2^{(n)}(x)} \right\}^p dx - \int_a^b r(x) \left\{ \frac{D_{*a}^\alpha w_1(x)}{D_{*a}^\alpha w_2(x)} \right\}^p dx$$

Moreover, we can extend these means, so in limiting cases for $p, q \neq 0, 1$

$$\lim_{q \rightarrow p} M_{p,q} = M_{p,p}$$

$$= \exp \left[\frac{\int_a^b Q_D(x) \left\{ \frac{w^{(n)}(x)^p}{w^{(n)}(x)^p} \right\} \log \left\{ \frac{w^{(n)}(x)}{w^{(n)}(x)} \right\} dx - \int_a^b r(x) \left\{ \frac{D_{*a}^\alpha w(x)^p}{D_{*a}^\alpha w(x)^p} \right\} \log \left\{ \frac{D_{*a}^\alpha w(x)}{D_{*a}^\alpha w(x)} \right\} dx}{\int_a^b Q_D(x) \left\{ \frac{w^{(n)}(x)^p}{w^{(n)}(x)^p} \right\} dx - \int_a^b r(x) \left\{ \frac{D_{*a}^\alpha w(x)^p}{D_{*a}^\alpha w(x)^p} \right\} dx} - \frac{2p-1}{p(p-1)} \right]$$

$$\lim_{p \rightarrow 0} M_{p,p} = M_{0,0}$$

$$= \exp \left[\frac{\int_a^b Q_D(x) \log^2 \left\{ \frac{w^{(n)}(x)}{w^{(n)}(x)} \right\} dx - \int_a^b r(x) \log^2 \left\{ \frac{D_{*a}^\alpha w(x)}{D_{*a}^\alpha w(x)} \right\} dx}{2 \int_a^b Q_D(x) \log \left\{ \frac{w^{(n)}(x)}{w^{(n)}(x)} \right\} dx - \int_a^b r(x) \log \left\{ \frac{D_{*a}^\alpha w(x)}{D_{*a}^\alpha w(x)} \right\} dx} + 1 \right]$$

$$\lim_{p \rightarrow 1} M_{p,p} = M_{1,1}$$

$$= \exp \left[\frac{\int_a^b Q_D(x) \left\{ \frac{w^{(n)}(x)}{w^{(n)}(x)} \right\} \log^2 \left\{ \frac{w^{(n)}(x)}{w^{(n)}(x)} \right\} dx - \int_a^b r(x) \left\{ \frac{D_{*a}^\alpha w(x)}{D_{*a}^\alpha w(x)} \right\} \log^2 \left\{ \frac{D_{*a}^\alpha w(x)}{D_{*a}^\alpha w(x)} \right\} dx}{2 \int_a^b Q_D(x) \left\{ \frac{w^{(n)}(x)}{w^{(n)}(x)} \right\} \log \left\{ \frac{w^{(n)}(x)}{w^{(n)}(x)} \right\} dx - \int_a^b r(x) \left\{ \frac{D_{*a}^\alpha w(x)}{D_{*a}^\alpha w(x)} \right\} \log \left\{ \frac{D_{*a}^\alpha w(x)}{D_{*a}^\alpha w(x)} \right\} dx} - 1 \right]$$

$$\lim_{q \rightarrow 0} M_{p,q} = M_{p,0}$$

$$= \exp \left[\frac{\int_a^b Q_D(x) \left\{ \frac{w^{(n)}(x)^p}{w^{(n)}(x)^p} \right\} dx - \int_a^b r(x) \left\{ \frac{D_{*a}^\alpha w(x)^p}{D_{*a}^\alpha w(x)^p} \right\} dx}{\left[2 \int_a^b Q_D(x) \log \left\{ \frac{w^{(n)}(x)}{w^{(n)}(x)} \right\} dx - \int_a^b r(x) \log \left\{ \frac{D_{*a}^\alpha w(x)^p}{D_{*a}^\alpha w(x)^p} \right\} dx \right]^{p-1}} \right]$$

$$= \exp \left[\frac{\left[\int_a^b Q_D(x) \left\{ \frac{w^{(n)}(x)^p}{w^{(n)}(x)^p} \right\} \log \left\{ \frac{w^{(n)}(x)}{w^{(n)}(x)} \right\} dx - \int_a^b r(x) \left\{ \frac{D_{*a}^\alpha w(x)^p}{D_{*a}^\alpha w(x)^p} \right\} \log \left\{ \frac{D_{*a}^\alpha w(x)}{D_{*a}^\alpha w(x)} \right\} dx \right]^{p-1}}{\int_a^b Q_D(x) \left\{ \frac{w^{(n)}(x)^p}{w^{(n)}(x)^p} \right\} dx - \int_a^b r(x) \left\{ \frac{D_{*a}^\alpha w(x)^p}{D_{*a}^\alpha w(x)^p} \right\} dx} \right]$$

4.2.2 Means for L_∞ fractional derivatives

We define means for L_∞ fractional derivative as $p, q \in \mathbb{R}$ where, $p, q \in \mathbb{R}$

$$\Pi_q = p(p-1) \int_a^b Q_L(x) \left\{ \frac{D_a^\beta w_1(x)}{D_a^\beta w_2(x)} \right\}^q dx - \int_a^b r(x) \left\{ \frac{D_a^\alpha w_1(x)}{D_a^\alpha w_2(x)} \right\}^q dx$$

and

$$\Pi_p = q(q-1) \int_a^b Q_L(x) \left\{ \frac{D_a^\beta w_1(x)}{D_a^\beta w_2(x)} \right\}^p dx - \int_a^b r(x) \left\{ \frac{D_a^\alpha w_1(x)}{D_a^\alpha w_2(x)} \right\}^p dx$$

In following we discuss limiting cases of above mean for $p, q \neq 0, 1$

$$\lim_{q \rightarrow p} M_{p,q} = M_{p,p}$$

$$= \exp \left[\frac{\int_a^b Q_L(x) \left\{ \frac{D_a^\beta w(x)^p}{D_a^\beta w(x)^p} \right\} \log \left\{ \frac{D_a^\beta w(x)}{D_a^\beta w(x)} \right\} dx - \int_a^b r(x) \left\{ \frac{D_a^\alpha w(x)^p}{D_a^\alpha w(x)^p} \right\} \log \left\{ \frac{D_a^\alpha w(x)}{D_a^\alpha w(x)} \right\} dx}{\int_a^b Q_L(x) \left\{ \frac{D_a^\beta w(x)^p}{D_a^\beta w(x)^p} \right\} dx - \int_a^b r(x) \left\{ \frac{D_a^\alpha w(x)^p}{D_a^\alpha w(x)^p} \right\} dx} - \frac{2p-1}{p(p-1)} \right]$$

$$\lim_{p \rightarrow 0} M_{p,p} = M_{0,0}$$

$$= \exp \left[\frac{\int_a^b Q_L(x) \log^2 \left\{ \frac{D_a^\beta w(x)}{D_a^\beta w(x)} \right\} dx - \int_a^b r(x) \log^2 \left\{ \frac{D_a^\alpha w(x)}{D_a^\alpha w(x)} \right\} dx}{2 \left[\int_a^b Q_L(x) \log \left\{ \frac{D_a^\beta w(x)}{D_a^\beta w(x)} \right\} dx - \int_a^b r(x) \log \left\{ \frac{D_a^\alpha w(x)}{D_a^\alpha w(x)} \right\} dx \right]} + 1 \right]$$

$$\lim_{p \rightarrow 1} M_{p,p} = M_{1,1}$$

$$= \exp \left[\frac{\int_a^b Q_L(x) \left\{ \frac{D_a^\beta w(x)}{D_a^\beta w(x)} \right\} \log^2 \left\{ \frac{D_a^\beta w(x)}{D_a^\beta w(x)} \right\} dx - \int_a^b r(x) \left\{ \frac{D_a^\alpha w(x)}{D_a^\alpha w(x)} \right\} \log^2 \left\{ \frac{D_a^\alpha w(x)}{D_a^\alpha w(x)} \right\} dx}{2 \int_a^b Q_L(x) \left\{ \frac{D_a^\beta w(x)}{D_a^\beta w(x)} \right\} \log \left\{ \frac{D_a^\beta w(x)}{D_a^\beta w(x)} \right\} dx - \int_a^b r(x) \left\{ \frac{D_a^\alpha w(x)}{D_a^\alpha w(x)} \right\} \log \left\{ \frac{D_a^\alpha w(x)}{D_a^\alpha w(x)} \right\} dx} - 1 \right]$$

$$\lim_{q \rightarrow 0} M_{p,q} = M_{p,0}$$

$$= \exp \left[\frac{\int_a^b Q_L(x) \left\{ \frac{D_a^\beta w(x)^p}{D_a^\beta w(x)^p} \right\} dx - \int_a^b r(x) \left\{ \frac{D_a^\alpha w(x)^p}{D_a^\alpha w(x)^p} \right\} dx}{2 \int_a^b Q_L(x) \log \left\{ \frac{D_a^\beta w(x)}{D_a^\beta w(x)} \right\} dx - \int_a^b r(x) \log \left\{ \frac{D_a^\alpha w(x)}{D_a^\alpha w(x)} \right\} dx} p(p-1) \right]^{\bar{p}}$$

$$\lim_{q \rightarrow 1} M_{p,q} = M_{p,1}$$

$$= \exp \left[\frac{\left[\int_a^b Q_L(x) \left\{ \frac{D_a^\beta w(x)}{D_a^\beta w(x)} \right\} \log \left\{ \frac{D_a^\beta w(x)}{D_a^\beta w(x)} \right\} dx - \int_a^b r(x) \left\{ \frac{D_a^\alpha w(x)}{D_a^\alpha w(x)} \right\} \log \left\{ \frac{D_a^\alpha w(x)}{D_a^\alpha w(x)} \right\} dx \right] p(p-1)}{\int_a^b Q_L(x) \left\{ \frac{D_a^\beta w(x)^p}{D_a^\beta w(x)^p} \right\} dx - \int_a^b r(x) \left\{ \frac{D_a^\alpha w(x)^p}{D_a^\alpha w(x)^p} \right\} dx} \right]^{\bar{p}}$$

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