wjert, 2020, Vol. 6, Issue 4, 298-307. *Review Article* ISSN 2454-695X

World Journal of Engineering Research and Technology

WJERT

www.wjert.org SJIF Impact Factor: 5.924

NEW METHOD WITH LAPLACE TRANSFORM TO SOLVE NONLINEAR PDES

Eltaib M. Abd Elmohmoud[1]* and Tarig M. Elzaki[2]

[1,2]Mathematics Department, College of Sciences and Arts, AlKamel, University of Jeddah, Saudi Arabia.

Article Received on 03/06/2020 **Article Revised on 24/06/2020** **Article Accepted on 14/07/2020**

***Corresponding Author**
Eltaib M. Abd Elmohmoud
Mathematics Department, College of Sciences and Arts, AlKamel, University of Jeddah, Saudi Arabia.

ABSTRACT

In this paper, we are going to introduce a brand new methodology and mix it with the mathematician remodel to unravel a number of non-linear partial differential equations. This method is characterized by the ease and abbreviation of many steps, as we have come up with an exact solution using only one step, depending on how to choose the appropriate preliminary approximation.

KEYWORDS: New Method, Laplace Transform, Non-Linear Partial Differential Equations, and Suitable Preliminary Approximation.

1. INTRODUCTION

It is well known that most of the phenomena that arise in mathematical physics and engineering fields can be described by linear or nonlinear partial differentials equations (PDEs).

The formulation of partial differentials equations and therefore the scientific interpretation of the models will not be discussed. It is to be noted that several methods are usually used in solving PDEs such that variational iteration method[1-13] integral transform method[14-22] and Projected Differential Transform Method.[14] The newly developed domain decomposition method and the related improvements of the modified technique and the noise terms phenomenon will be effectively used. In this paper, we introduce a simple new method with Laplace transform to unravel nonlinear partial differentials equations

(PDEs). This method is formally proved to provide the solution in terms of a rapidly convergent infinite series that may yield the exact solution in many cases.

2. The New Method

To display the new method for solving nonlinear partial differential equations, we consider the equation,

$$L_r w(r,\theta) + L_\theta w(r,\theta) + \mathrm{R}(w(r,\theta)) + \mathrm{F}(w(r,\theta)) = \mathrm{g}(r,\theta)$$

(1)

Where L_r , is the highest order differential (here we assume that $L_r = \frac{\partial^2}{\partial r^2}$) in r, L_θ is the highest order differential in θ, R contains the remaining linear terms of lower derivatives, $\mathrm{F}(w(r,\theta))$ is an analytic nonlinear term, and $\mathrm{g}(r,\theta)$ is an inhomogeneous or forcing term.

To find the solution of equation (1), apply Laplace transform to both sides of (1) gives

$$l\,\{L_r w(r,\theta)\} = l\,[\mathrm{g}(r,\theta) - L_\theta w(r,\theta) - Rw(r,\theta) - \mathrm{F}\,w(r,\theta)]$$

(2)

By taking the inverse Laplace transform, we obtain:

$$w(r,\theta) = w(0,\theta) + rw_r(0,\theta) + l^{-1}\tfrac{1}{s^2}[l\,g(r,\theta)] - l^{-1}\tfrac{1}{s^2}\{l\,[L_\theta w(r,\theta) + \mathrm{R}\,w(r,\theta) + \mathrm{F}\,w(r,\theta)]\}$$

(3)

We proceed in the same manner by calculating the solution $w(r,\theta)$ in a series form:

$$w(r,\theta) = \sum_{n=0}^{\infty} w_n(r,\theta)$$

(4)

Then equation (3) becomes,

$$\sum_{\mathrm{n}=0}^{\infty} \mathrm{w}_\mathrm{n}(\mathrm{r},\theta) = \mathrm{w}(0,\theta) + \mathrm{rw}_\mathrm{r}(0,\theta) + l^{-1}\tfrac{1}{\mathrm{s}^2}[l\,g(r,\theta)]$$
$$- l^{-1}\tfrac{1}{\mathrm{s}^2}\{l\,[L_\theta w_n(r,\theta) + \mathrm{R}\,w_n(r,\theta) + Fw_n(r,\theta)]\}$$

(5)

The components $w_n(r,\theta)$, $\mathrm{n} \geq 0$ of the solution $w(r,\theta)$ can be recursively determined by using the relation,

$$\mathrm{w}_0(\mathrm{r},\theta) = \mathrm{w}(0,\theta) + \mathrm{rw}_\mathrm{r}(0,\theta) + l^{-1}\tfrac{1}{\mathrm{s}^2}l\,[g(r,\theta)]$$

$$w_{n+1}(r,\theta) = -l^{-1}\{\tfrac{1}{\mathrm{s}^2}l\,[L_\theta w_n(r,\theta) + \mathrm{R}\,w_n(r,\theta) + Fw_n(r,\theta)]\} \qquad \mathrm{n} \geq 0$$

(6)

The first few components can be identified by:

$$\mathrm{w}_0(\mathrm{r},\theta) = \mathrm{w}(0,\theta) + \mathrm{rw}_\mathrm{r}(0,\theta) + l^{-1}\tfrac{1}{s^2} l\left[g(r,\theta)\right]$$

$$w_1(r,\theta) = -l^{-1}\left\{\tfrac{1}{s^2} l\left[L_\theta w_0(r,\theta) + \mathrm{R}\, w_0(r,\theta) + F w_0(r,\theta)\right]\right\}$$

$$w_2(r,\theta) = -l^{-1}\left\{\tfrac{1}{s^2} l\left[L_\theta w_1(r,\theta) + \mathrm{R}\, w_1(r,\theta) + F w_1(r,\theta)\right]\right\}$$

.

.

.

And the solution in a series form is readily obtained by using Eq. (4).

3. Application

In the following, several distinct nonlinear partial differential equations will be discussed to illustrate the procedure outlined above.

Example 1

Consider the nonlinear partial differential equation,

$$w_t + w w_r = r + rt^2, \quad \mathrm{w(r,0)=0} \quad , \mathrm{t>0} \qquad (7)$$

Applying Laplace transform Eq. (7), to find:

$$s l\, w(\mathrm{r,t}) - \mathrm{w(r,0)} = \frac{r}{s} + l\left[rt^2 - w(\mathrm{r,t}) w_r(\mathrm{r,t})\right]$$

Or $l\, w(\mathrm{r,t}) - \mathrm{w(r,0)} = \dfrac{r}{s^2} + \dfrac{1}{s} l\left[rt^2 - w(\mathrm{r,t}) w_r(\mathrm{r,t})\right]$

(8)

Applying the inverse Laplace transform to Eq. (8) to obtain:

$$w(\mathrm{r,t}) = rt + l^{-1}\tfrac{1}{s}\left[rt^2 - w(\mathrm{r,t}) w_r(\mathrm{r,t})\right]$$

The recursive relation is,

$$w_{n+1}(r,t) = l^{-1}\tfrac{1}{s}\left\{l\left[rt^2 - w_n(\mathrm{r,t})(\mathrm{w})_n(\mathrm{r,t})\right]\right\}$$

(9)

$$w_0(\mathrm{w,t}) = rt$$

The first few components are given by:

$$w_0(\mathrm{w,t}) = rt$$

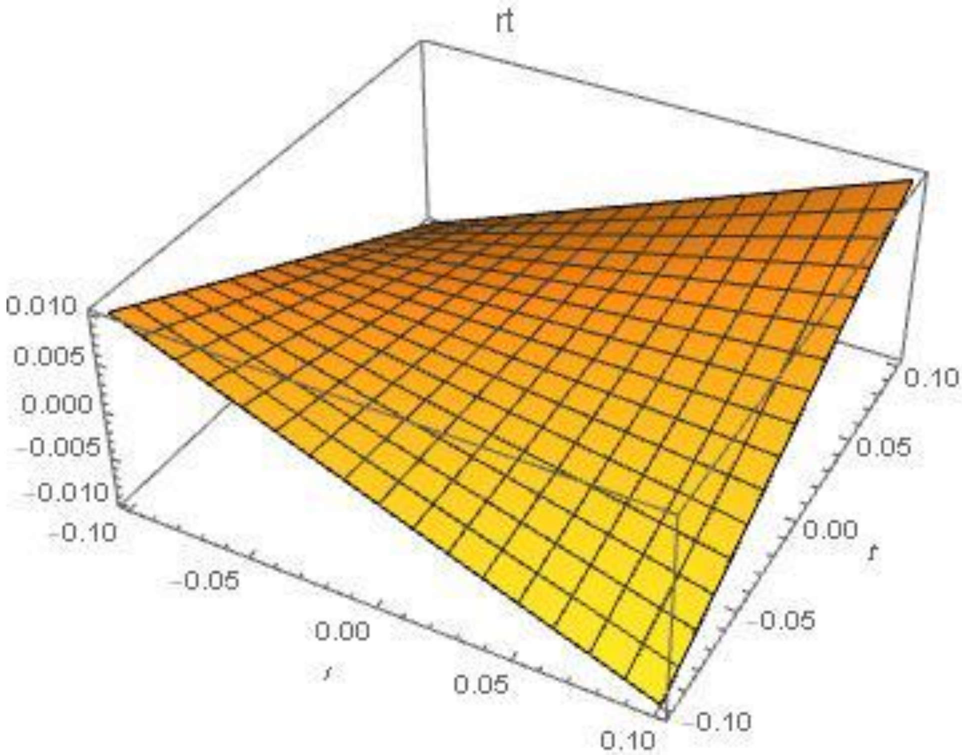
$$w_1(r,t) = l^{-1}\tfrac{1}{s}\left\{l\left[rt^2 - rt^2\right]\right\} = 0$$

(10)

$$w_{n+2}(r,t) = 0 \ , \quad n \geq 0$$

In view of Eq. (10) the exact solution is given by:

$$w(r,t) = rt$$



Example 2

Consider the second order nonlinear partial differential equation,

$$w_{rr} - w_r w_{\theta\theta} = -r + w, \quad w(0,\theta) = \sin\theta, \quad w_r(0,\theta) = 1$$

(11)

Applying Laplace transform to Eq. (11) and making use of the initial conditions gives,

$$l\,w(\mathrm{r,t}) = \frac{\sin\theta}{s} + \frac{1}{s^2} - \frac{1}{s^4} + \tfrac{1}{s^2} l\left[w(\mathrm{r,t}) + w_r(\mathrm{r,t}) w_{\theta\theta}(\mathrm{r,t})\right]$$

(12)

Proceeding as before, we find,

$$w(\mathrm{r},\theta) = \sin\theta + \mathrm{r} - \frac{r^3}{3!} + l^{-1}\tfrac{1}{s^2}\left\{l\left[w(\mathrm{r,t}) + w_r(\mathrm{r,t}) w_{\theta\theta}(\mathrm{r,t})\right]\right\}$$

(13)

To use the new method, we identify the component w_0 by $\mathrm{w}_0(\mathrm{r},\theta) = \mathrm{sinr} + \mathrm{r}$

And the remaining term $-\frac{r^3}{3!}$ will be assigned to $w_1(\mathrm{r},\theta)$ among other terms, and then we obtain the recursive relation,

$$w_0(\mathrm{r},\theta) = \sin\theta + r$$

$$w_1(r,\theta) = -\frac{r^3}{3!} + 1^{-1}{}_{\frac{1}{s^2}}\left\{1\left[w_0(\mathrm{r,t}) + w_{0r}(\mathrm{r,t})w_{0\theta\theta}(\mathrm{r,t})\right]\right\}$$

(14)

$$w_{n+1}(r,\theta) = 1^{-1}{}_{\frac{1}{s^2}}\left\{1\left[w_n(\mathrm{r,t}) + w_{nr}(\mathrm{r,t})w_{n\theta\theta}(\mathrm{r,t})\right]\right\}, \qquad \mathrm{n} \geq 1$$

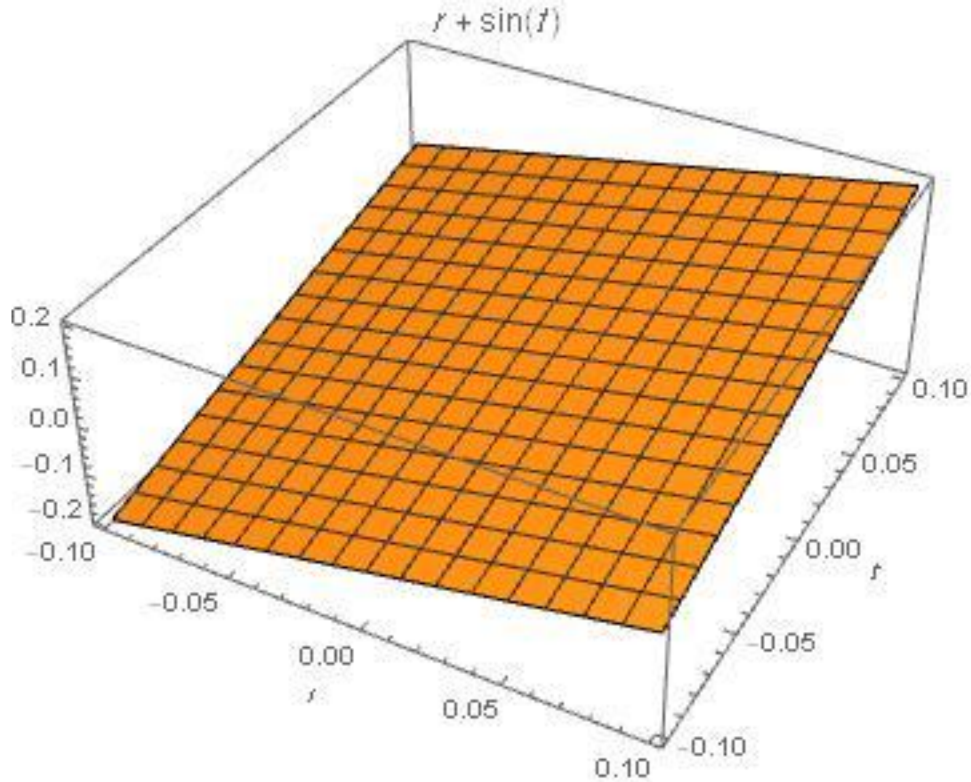
Consequently we obtain,

$$w_0(\mathrm{r},\theta) = \sin\theta + r$$

$$w_1(r,\theta) = -\frac{r^3}{3!} + 1^{-1}{}_{\frac{1}{s^2}}\left\{1\left[\sin\theta + r - \sin\theta\right]\right\} = -\frac{r^3}{3!} + \frac{r^3}{3!} = 0$$

Then the exact solution is given by:

$$w(\mathrm{r},\theta) = \sin\theta + r$$



Example 3

Consider the nonlinear PDE,

$$w_{rr} + \tfrac{1}{4} w_{\theta}^{2} = w, \quad \mathrm{w(0,w)}=1+\theta^{2}, \qquad \mathrm{w}_{r}(0,\theta) = 1$$

(15)

Appling Laplace transform to Eq. (15) and using the given conditions we find:

$$\mathrm{l}\, w(\mathrm{r},\theta) = \tfrac{1+\theta^{2}}{s} + \tfrac{1}{s^{2}} + \tfrac{1}{s^{2}}\mathrm{l}\left[\mathrm{w}(\mathrm{r},\theta) - \tfrac{1}{4}\mathrm{w}_{\theta}^{2}(\mathrm{r},\theta)\right]$$

(16)

Proceeding as before, we obtain

$$\sum_{n=0}^{\infty} w_{n}(\mathrm{r},\theta) = 1 + \theta^{2} + r + \mathrm{l}^{-1}\tfrac{1}{s^{2}}\left\{\mathrm{l}\left[w_{n}(\mathrm{r},\theta) - \tfrac{1}{4} w_{n\theta}^{2}(\mathrm{r},\theta)\right]\right\}$$

The new method admits the use of the recursive relation,

$$w_{0}(\mathrm{r},\theta) = 1 + \theta^{2} + r$$

$$w_{n+1}(\mathrm{r},\theta) = \mathrm{l}^{-1}\tfrac{1}{s^{2}}\left\{\mathrm{l}\left[w_{n}(\mathrm{r},\theta) - \tfrac{1}{4} w_{n\theta}^{2}(\mathrm{r},\theta)\right]\right\} \qquad \mathrm{n} \geq 0$$

(17)

The first few components of the solution $w(r,\theta)$ are given by:

$$w_{0}(\mathrm{r},\theta) = \theta^{2} + 1 + r$$

$$w_{1}(\mathrm{r},\theta) = \mathrm{l}^{-1}\tfrac{1}{s^{2}}\left\{\mathrm{l}\left[\theta^{2} + r + 1 - \theta^{2}\right]\right\} = \mathrm{l}^{-1}\left[\tfrac{1}{s^{3}} + \tfrac{1}{s^{4}}\right] = \tfrac{r^{2}}{2!} + \tfrac{r^{3}}{3!}$$

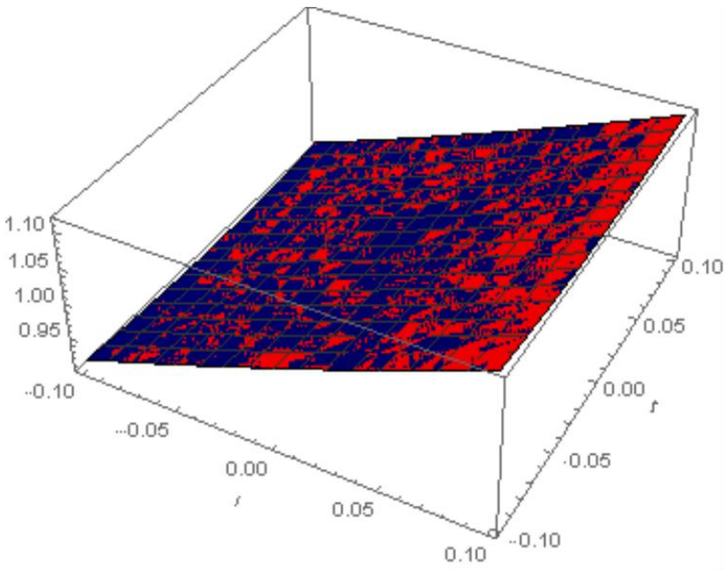
$$w_{2}(\mathrm{r},\theta) = \mathrm{l}^{-1}\tfrac{1}{s^{2}}\left\{\mathrm{l}\left[\tfrac{r^{2}}{2!} + \tfrac{r^{3}}{3!}\right]\right\} = \tfrac{r^{4}}{4!} + \tfrac{r^{5}}{5!}$$

And so on, for the other components, consequently, the solution in a series form is given by

$$w\,(\mathrm{r},\theta) = \theta^{2} + (1 + \mathrm{r} + \tfrac{r^{2}}{2!} + \tfrac{r^{3}}{3!} + \ldots\ldots)$$

Which is gives the solution in a closed form as

$$w(\mathrm{r},\theta) = \theta^{2} + e^{r}$$



Example 4

$$w_{rr} + w^2 - w_\theta^2 = 0, \quad \mathrm{w}(0,\theta)=0, \qquad \mathrm{w}_r(0,\theta) = e^\theta,$$

(18)

Using the same steps that we use as before to find:

$$s^2 \mathrm{l}\, w(\mathrm{r},\theta) - e^\theta = \mathrm{l}\left[w_\theta^2(\mathrm{r},\theta) - w^2(\mathrm{r},\theta) \right]$$

$$\mathrm{l}\, w(\mathrm{r},\theta) = \tfrac{e^\theta}{s^2} + \tfrac{1}{s^2} \mathrm{l}\left[w_\theta^2(\mathrm{r},\theta) - w^2(\mathrm{r},\theta) \right]$$

$$w(\mathrm{r},\theta) = \mathrm{r}\, e^\theta + \mathrm{l}^{-1} \tfrac{1}{s^2}\left\{ \mathrm{l}\left[w_\theta^2(\mathrm{r},\theta) - w^2(\mathrm{r},\theta) \right]\right\}$$

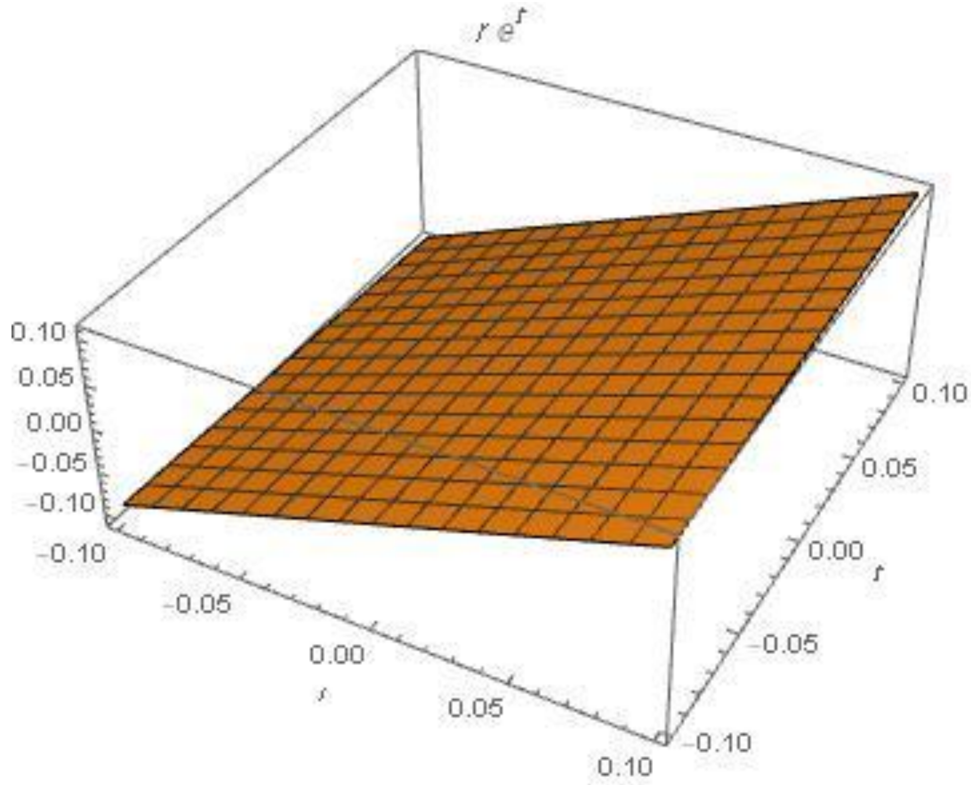
$$w_0(\mathrm{r},\theta) = \mathrm{r}\, e^\theta$$

$$w_{n+1}(\mathrm{r},\theta) = \mathrm{l}^{-1} \tfrac{1}{s^2}\left\{ \mathrm{l}\left[w_{n\theta}^2(\mathrm{r},\theta) - w_n{}^2(\mathrm{r},\theta) \right]\right\}$$

$$w_1 = 0, \text{ ....}$$

And then the exact solution is:

$$w(\mathrm{r},\theta) = re^\theta$$



CONCLUSION

In this paper, a number of non-linear partial differential equations were solved during a new way using Laplace transform. Where it had been clarified the way to choose the first approximation that results in a particular solution. We found that this new method is extremely effective in solving non-linear partial differential equations because it has simple and really fast in reaching a particular solution.

Availability of data and materials

Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

Competing interests: The authors declare that they have no competing interests.

Authors' contributions: The authors read and agreed the final manuscript.

REFERENCES

1. J. Biazar, H. Ghazvini, He's variational iteration method for solving linear and non-linear systems of ordinary differential equations, Appl. Math. Comput, 2007; 191: 287–297.
2. J.H. He, Variational iteration method for delay differential equations, Commun.Nonlinear Sci. Numer. Simul, 1997; 2(4): 235–236.

3. J.H. He, Variational iteration method—a kind of non-linear analytical technique: some examples, Int. J. Nonlinear Mech, 1999; 34: 699–708.
4. J.H. He, Variational iteration method for autonomous ordinary differential systems, Appl. Math. Comput, 2000; 114: 115–123.
5. J.H. He, X.H. Wu, Variational iteration method: new development and applications, Comput. Math. Appl, 2007; 54: 881–894.
6. S.A. Khuri, A. Sayfy, A Laplace variational iteration strategy for the solution of differential equations, *Applied Mathematics Letters*, 2012; 25: 2298–2305.
7. E. Hesameddini and H. Latifizadeh, "Reconstruction of variational iteration algorithms using the Laplace transform," *International Journal of Nonlinear Sciences and Numerical Simulation*, 2009; 10: 11-12, pp. 1377–1382.
8. G. C. Wu, D. Baleanu, "Variational iteration method for fractional calculus - a universal approach by Laplace transform," *Advances in Difference Equations*, 2013; 18-27, 2013.
9. X. J. Yang, D. Baleanu, "Fractal heat conduction problem solved by local fractional variation iteration method," *Thermal Science*, 2012, Doi: 10.2298/TSCI121124216Y.
10. G.C. Wu, Variational iteration method for solving the time-fractional diffusion equations in porous medium, Chin. Phys. B, 2012; 21: 120504.
11. G.C. Wu, D. Baleanu, Variational iteration method for the Burgers' flow with fractional derivatives-New Lagrange multipliers, Applied Mathematical Modelling, 2012; 37: 6183–6190.
12. G.C. Wu, Challenge in the variational iteration method-a new approach to identification of the Lagrange mutipliers, Journal of King Saud University-Science, 2013; 25: 175-178.
13. G.C. Wu. Laplace transform Overcoming Principle Drawbacks in Application of the Variational Iteration Method to Fractional Heat Equations, THERMAL SCIENCE, 2012; 16(4): 1257-1261.
14. Tarig M. Elzaki, Application of Projected Differential Transform Method on Nonlinear Partial Differential Equations with Proportional Delay in One Variable, World Applied Sciences Journal, 2014; 30(3): 345-349. DOI: 10.5829/idosi.wasj.2014.30.03.1841.
15. Tarig M. Elzaki, andJ. Biazar, Homotopy Perturbation Method and Elzaki Transform for Solving System of Nonlinear Partial Differential Equations, World Applied Sciences Journal, 2013; 24(7): 944-948. DOI: 10.5829/idosi.wasj.2013.24.07.1041.
16. Tarig. M. Elzaki- Salih M. Elzaki –Elsayed A. Elnour, On the New Integral Transform "ELzaki Transform" Fundamental Properties Investigations and Applications, Global

Journal of Mathematical Sciences: Theory and Practical. ISSN 0974-3200, 2012; 4(1): 1-13 © International Research Publication House.

17. Tarig M. Elzaki, and Salih M. Elzaki, On the Connections Between Laplace and Elzaki Transforms, Advances in Theoretical and Applied Mathematics. ISSN 0973-4554, 2011; 6(1): 1-11.
18. Djelloul Ziane, Tarig M. Elzaki and Mountassir Hamdi Cherif, Elzaki transform combined with variational iteration method for partial differential equations of fractional order, Fundamental Journal of Mathematics and Applications, 2018; 1(1): 102-108.
19. Aisha Abdullah Alderremy, Tarig M. Elzaki, Mourad Chamekh, New transform iterative method for solving some Klein-Gordon equations, Results in Physics, doi: https://doi.org/10.1016/j.rinp.2018.07.004, September 2018; 10: 655-659.
20. Djelloul Ziane, Tarig M. Elzaki and Mountassir Hamdi Cherif, Elzaki transform combined with variational iteration method for partial differential equations of fractional order, Fundamental Journal of Mathematics and Applications, 2018; 1(1): 102-108. Journal Homepage: www.dergipark.gov.tr/fujma.
21. M. DUZ, T. M. ELZAKI, SOLUTION OF CONSTANT COEFFIENTS PARTIAL DERIVATIVE EQUATIONS WITH ELZAKI TRANSFORM METHOD, TWMS J. App. Eng. Math, 2019; 9(3): 563-570.
22. Tarig M. Elzaki and A.A. Ishag, Modified Laplace Transform and Ordinary Differential Equations with Variable Coefficients, World Engineering & Applied Sciences Journal, 2019; 10(3): 79-84. DOI: 10.5829/idosi.weasj.2019.79.84.