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EXPANDED FAMILIES OF ORTHOGONAL BINARY HADAMARD MATRICES

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ABSTRACT

This paper considers new families of orthogonal binary matrices which are similar to orthogonal binary Walsh-Hadamard matrices. It is shown that beyond Walsh-Hadamard matrices, additional orthogonal binary square matrices of size $N = 2^n$ ($n = 3, 4, 5, \ldots$) exist. The total amount of these binary orthogonal matrices increases quickly with an increasing value of N. This paper presents the matrix formation rules for these families. The results of this paper apply to CDMA communication and telemetric systems.

KEYWORDS: Binary orthogonal matrices, binary signals, binary sequences, CDMA, Walsh-Hadamard matrices.

INTRODUCTION

Orthogonal binary functions were first introduced by J. L. Walsh in 1923.[1] These orthogonal binary functions are widely used in discrete signals theory.[2],[3] In practice, binary signals utilizing Walsh functions are used in CDMA wireless communication systems.[4],[9]

As is well known, orthogonal binary Walsh functions exist only for $N = 2^n$, where n = 1, 2, 3, …. Walsh functions can be represented as Walsh-Hadamard (WH) matrices H_N,

$$H_N = \begin{bmatrix} H_{N/2} & H_{N/2} \\ H_{N/2} & -H_{N/2} \end{bmatrix}, \tag{1}$$

Where $H_{N=1} = [+]$ is the single or elementary matrix:

N=1

$H_1 = [+]$.

For $N = 2^n$ (n = 1, 2), WH matrices H_N (1) have the following form:

N=2 (n = 1)

$$H_2 = \begin{bmatrix} H_1 & H_1 \\ H_1 & -H_1 \end{bmatrix} = \begin{bmatrix} + & + \\ + & - \end{bmatrix},$$

N = 4 (n = 2)

$$H_4 = \begin{bmatrix} H_2 & H_2 \\ H_2 & -H_2 \end{bmatrix} = \begin{bmatrix} + & + & + & + \\ + & - & + & - \\ + & + & - & - \\ + & - & - & + \end{bmatrix} \begin{matrix} G\downarrow \\ G_1 \\ G_2 \\ \} G_3. \end{matrix} \qquad (2)$$

It is also possible to represent the same WH matrices in another way, namely,

$$H_N = \begin{bmatrix} H_{N/2} & H_{N/2} \\ H_{N/2} & -H_{N/2} \end{bmatrix} = H_2 \otimes H_2 \otimes \ \dots H_2 \otimes H_1 = (H_2 \otimes)^n \otimes H_1 = (H_2 \otimes)^n, \qquad (3)$$

Where matrix H_2 is the Hadamard matrix of order 2 (N=2) (2)

$$H_2 = \begin{bmatrix} H_1 & H_1 \\ H_1 & -H_1 \end{bmatrix} = \begin{bmatrix} + & + \\ + & - \end{bmatrix}.$$

The symbol "$\otimes$" is called the symbol of Kronecker multiplication (Kronecker product) and is defined in.[5] In equation 3, the notation $(H_2 \otimes)^n$ means applying the Kronecker multiplication procedure n times.

The matrix H_N (3) is formed in a step-by-step (n steps) process of Kronecker multiplications of the elementary matrix H_1 by the Hadamard order 2 matrix H_2 (2). In matrix H_N, the sequence length corresponds to the row length, and the number of rows corresponds to the number of sequences. During this procedure (3), the sequence length and the number of rows of matrices H_N increase by a factor of two after each Kronecker multiplication by matrix H_2. Orthogonal binary sequences of WH matrices consist of sequences of plus and minus signs, i.e., sequences of +1 and -1, of length $N = 2^n$ (n = 1, 2, 3, 4, …). The length of the sequences defines the length of the matrix rows. And the total number of binary sequences, i.e., the number of rows, is also equal to $N = 2^n$.

Notice that as the integer N increases, the number of different binary sequences of length N grows faster than N. The result means that for any value of N, there are some possibilities to create other square N x N matrices with distinguished properties.

This paper will show that for any $N = 2^n$ (n = 3, 4, 5, …), additional orthogonal binary matrices exist with size N x N and similar properties to WH matrices. The total number of binary orthogonal matrices grows at a rate of $N = 2^n$ and equals $Q_N = 2^{(N-(n+1))}$, where Q_N is the total amount of orthogonal binary square matrices for $N = 2^n$ (n = 3, 4, 5, …) including WH matrices. These matrices can be called an expanded family of binary orthogonal Hadamard matrices because all are related to Hadamard matrices for N = 4. The paper subsequently presents the formation rules for these matrices.

The results of this paper apply to CDMA communication and telemetric systems.[5]

FORMATION OF BINARY SEQUENCES

All binary sequences $A_{N,i}$ of pluses and minuses of length N, where N is positive integers, can be represented in matrix form as

$$M_{N,m} = \left[A_{N,i}\right] = \begin{bmatrix} A_{N,1} \\ A_{N,2} \\ \cdots \\ A_{N,m} \end{bmatrix} \begin{matrix} k\downarrow \\ 0 \\ 1 \\ \ldots \\ m-1 \end{matrix} = \begin{bmatrix} + & + & + & + & \ldots & \ldots \\ + & + & - & + & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ + & - & + & + & \ldots & \ldots \end{bmatrix} \begin{matrix} k\downarrow \\ 0 \\ 1 \\ \ldots \\ m-1. \end{matrix} \qquad (4)$$

$A_{N,i}$ are the binary sequences (i = 1, 2, 3, …, m), N is the length of the binary sequences (number of matrix columns), m is the total number of different binary sequences with length N, and k is the ordering number of the matrix $M_{N,m}$ row (k = 0, 1, 2, ..., m-1).

When constructing a matrix $M_{N,m}$ (4) for any N (N = 2, 3, 4, 5, …), it is possible to use the most straightforward rule, namely,

$$M_{N,m} = \begin{bmatrix} M_{(N-1),m} & + \\ M_{(N-1),m} & - \end{bmatrix}. \qquad (5)$$

By this rule, matrix $M_{N,m}$ consists of two parts: an upper part and a lower part. The upper part contains all matrix $M_{(N-1),m}$ rows and a plus sign at the end of each row. The lower part contains all matrix $M_{(N-1),m}$ rows and a minus sign at the end of each row. The matrix $M_{(N-1),m}$ to matrix $M_{N,m}$ transformation procedure increases the length of sequences (length of the rows) by one element, but the total number of sequences (number of rows) doubles. Notably, rule (5) provides the ability to obtain all the different binary sequences of length N, distinguished by at least one element or the order of element sequences.

For N = 1, we have the simple case:

$$M_{N=1,m} = [+] = H_1. \qquad (6)$$

Thus, the N = 1 matrix $M_{N,m}$ (4) consists of one row with one element, namely, sign "+". Matrix $M_{N=1,m}$ is the elementary matrix; i.e., for N = 1, $M_{N=1,m} = H_1$ (1), (2).

The following, beginning with N = 2, are examples of this rule (5).

For N = 2, per rule (5), we have the following:

$$M_{N=2,m} = \begin{bmatrix} M_{N=1,m} & + \\ M_{N=1,m} & - \end{bmatrix} \begin{matrix} k\downarrow \\ 0 \\ 1, \end{matrix}$$

or

$$M_{N=2,m} = \begin{bmatrix} + & + \\ + & - \end{bmatrix} \begin{matrix} k\downarrow \\ 0 \\ 1. \end{matrix} \qquad (7)$$

Thus, for N = 2, matrix $M_{N,m}$ (4) consists of two rows (k = 0, k = 1) with two elements in each row, and the number of different binary sequences m equals 2. And matrix $M_{N=2,m}$ is the orthogonal binary WH matrix; i.e., $M_{N=2,m} = H_2$ (1), (2).

For N = 3, per (5), we have the following:

$$M_{N=3,m} = \begin{bmatrix} M_{N=2,m} & + \\ M_{N=2,m} & - \end{bmatrix} = \begin{bmatrix} + & + & + \\ + & - & + \\ + & + & - \\ + & - & - \end{bmatrix} \begin{matrix} k\downarrow \\ 0 \\ 1 \\ 2 \\ 3. \end{matrix} \qquad (8)$$

For N = 4, per (5), we have the following:

$$M_{N=4,m} = \begin{bmatrix} M_{N=3,m} & + \\ M_{N=3,m} & - \end{bmatrix} = \begin{bmatrix} + & + & + & + \\ + & - & + & + \\ + & + & - & + \\ + & - & - & + \\ + & + & + & - \\ + & - & + & - \\ + & + & - & - \\ + & - & - & - \end{bmatrix} \begin{matrix} k\downarrow \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7. \end{matrix} \qquad (9)$$

Thus, for N = 4, there are eight different binary sequences with length N = 4. And the sequences with k = 0, k = 5, k = 6, and k = 3 (9) are the sequences of the WH orthogonal matrix for N = 4 (1), (2). Using the same rule (5), one can construct matrices $M_{N,m}$ for any value of N, where N is a positive integer.

Equations (6), (7), (8), and (9) illustrate that the number of different binary sequences m_N grows much faster than N. By analyzing the matrices $M_{N,m}$ construction procedure for N = 2, 3, and 4 (7) (8) (9), one may observe that rule (5) may be used to form matrices $M_{N,m}$ that include all of the possible different binary sequences for the current value of N. At least one element distinguishes the different binary sequences or they are distinguished by the order of elements inside the sequences. As a function of N, the total value of these different sequences is defined as $m_N = f(N) = 2^{N-1}$. Notice that the value $m_N = 2^{N-1}$ corresponds to the binary sequences with the sign "+" as the first element of the sequences.

For binary sequences where the first element is a plus, for any integer N, there usually exist "mirror" alternative sequences where all the signs of matrix sequences $A_{N,i}$ are changed to the opposite signs, i.e., plus signs replace the minus signs and minus signs replace the plus signs.

In this paper, we only use the sequences where the plus sign is the first element. In this scenario, the maximum number of different sequences equals $m_N = f(N) = 2^{N-1}$ for any N where N is a positive integer.

As detailed above, it is possible to build matrices $M_{N,m}$ using rule (5) for any value of N (N = 2, 3, 4, 5, ….). Because the total amount of binary sequences $m_N = 2^{N-1}$ increases so fast with N (for example, for N= 4, 5, 6, 7, and 8, the values of m_N are 8, 16, 32, 64, and 128 respectively), the matrix construction procedure following rule (5) is time-consuming. And after the construction of matrix $M_{N,m}$, it is necessary to find sequences inside the matrix that have unique properties (e.g., to find the orthogonality of the sequences). This procedure is quite time-consuming.

This paper considers an alternative approach for constructing binary sequences for $N = 2^n$ (n = 2, 3, 4, 5, …). The alternate approach simplifies the construction procedure of matrix $M_{N,m}$ with the automatic separation of all sequences on orthogonal binary matrices with the size N x N. This approach is based on matrix circular rotation procedures (Appendix 1).

ORTHOGONAL BINARY MATRICES

It is possible to simplify the construction procedure of matrices $M_{N,m}$ (5) for $N = 2^n$ by using Kronecker multiplication (3) by following a circular rotation of the matrix sequences (Appendix 1).

Only two sequences exist with the length N = 2 (7) and matrix $M_{N,m}$ is a WH orthogonal binary matrix, i.e., $M_{N=2,m} = H_{N=2}$ (1), (2).

For N = 4 (9), instead of using rule (5), it is possible to perform a Kronecker multiplication of matrix $M_{N=2,m}$ by matrix H_2 (3) followed by a circular rotation of the matrix sequences. After the Kronecker multiplication of matrix $M_{N=2,m}$ by matrix H_2, we obtain:

$$H_{N=4} = H_2 \otimes M_{N=2,m} = \begin{bmatrix} M_{N=2,m} & M_{N=2,m} \\ M_{N=2,m} & -M_{N=2,m} \end{bmatrix} = \begin{bmatrix} H_2 & H_2 \\ H_2 & -H_2 \end{bmatrix} = \begin{bmatrix} + & + & + & + \\ + & - & + & - \\ + & + & - & - \\ + & - & - & + \end{bmatrix} \begin{matrix} k\downarrow \\ 0 \\ 1 \\ 2 \\ 3. \end{matrix} \quad (10)$$

Matrix $H_{N=4}$ is a WH matrix for N = 4 (1), (2).

Then, using the synchronized circular rotation procedure on the right half of matrix $H_{N=4}$ (10) on step R = 1 (Appendix 1), we obtain one more orthogonal matrix:

$$H_{N=4}^{(\uparrow R/R=1)} = \begin{bmatrix} H_2 & H_2^{(\uparrow R=1)} \\ H_2 & -H_2^{(\uparrow R=1)} \end{bmatrix} = \begin{bmatrix} + & + & + & - \\ + & - & + & + \\ + & + & - & + \\ + & - & - & - \end{bmatrix} \begin{matrix} k\downarrow \\ 0 \\ 1 \\ 2 \\ 3. \end{matrix} \quad (11)$$

Notice that matrix $M_{N=4,m}$ (9) consists of the identical eight binary sequences as in matrices (10) and (11).

For convenience, matrices (10) and (11) will be enumerated. Matrix $H_{N=4}$ will be enumerated as matrix number one and marked as $H_{N=4}^{(1)}$. Matrix $H_{N=4}^{(\uparrow R/R=1)}$ will be enumerated as matrix number two and marked as $H_{N=4}^{(2)}$ for N=4. I.e.,

$$H_{N=4}^{(1)} = H_{N=4}, \qquad H_{N=4}^{(2)} = H_{N=4}^{(\uparrow R/R=1)}. \quad (12)$$

Matrix number one $H_{N=4}^{(1)}$ is the Walsh-Hadamard matrix H_N for N = 4 (1), (2). Matrix number two $H_{N=4}^{(2)}$ is identical to the orthogonal Hadamard back-circulant matrix for N = 4.[6] Thus, matrices (10) and (11) are different types of Hadamard binary orthogonal matrices for N = 4.

Notice that both matrices, $H_{N=4}^{(1)}$ and $H_{N=4}^{(2)}$ (10), (11), and (12), are independent orthogonal binary matrices. By definition of independent orthogonal binary matrices, this result means that the binary sequences inside each of these binary matrices are orthogonal, but the binary sequences belonging to other matrices are not orthogonal. As above, for N=4, only eight ($m_N = 2^{N-1} = 8$) different binary sequences exist with different elements or different orders of pluses and minuses inside the sequences (9). Except for these eight sequences, no other binary sequences exist for N=4. And as shown, all of these eight binary sequences can be divided into two independent orthogonal matrices $H_{N=4}^{(1)}$ and $H_{N=4}^{(2)}$ of size N x N (10), (11), and (12). The orthogonal matrices $H_{N=4}^{(1)}$ and $H_{N=4}^{(2)}$ consist only of different sequences; i.e., these matrices have no identical sequences.

Thus, by using the Kronecker multiplication of matrix H_2 by matrix $M_{N=2,m}$ (10) and following a synchronized rotation procedure of matrix sequences (11) (Appendix 1), it is possible to obtain two independent orthogonal square matrices for N = 4 (10), (11) and (12).

In this case, the procedure automatically separates matrix $M_{N=4,m}$ (9) into two orthogonal binary square matrices with size N = 4. These are matrix number one $H_{N=4}^{(1)}$, and matrix number two $H_{N=4}^{(2)}$.

Orthogonal binary matrices $H_{N=4}^{(1)}$ and $H_{N=4}^{(2)}$ have different properties. Binary signals corresponding to unique groups (groups G_1, G_2, G_3) (2) of matrix number one $H_{N=4}^{(1)}$ sequences, i.e., WH matrix sequences, have zero cross correlation properties that apply to periodic signals.[8] Orthogonal binary sequences of matrix two $H_{N=4}^{(2)}$, i.e., the Hadamard back-circulant matrix for N = 4, do not have this property. However, the sequences of matrix $H_{N=4}^{(2)}$ do have other interesting properties. The first property is that aperiodic binary signals corresponding to these sequences are Barker coded signals for N = 4.[7] And second property is that autocorrelation functions of periodic signals corresponding to all these sequences are perfect.[4]

Consider the case when N = 8. There are several different ways to construct binary sequences for N = 8, including (5).

According to the following steps, we will use an approach to progress from N = 4 to N = 8.

1. In the **first step**, perform the Kronecker multiplication of matrices $H_{N=4}^{(1)}$ and $H_{N=4}^{(2)}$ (10), (11), and (12) by matrix H_2:

$$H_{N=8}^{(1)} = H_2 \otimes H_{N=4}^{(1)} = \begin{bmatrix} H_{N=4}^{(1)} & H_{N=4}^{(1)} \\ H_{N=4}^{(1)} & -H_{N=4}^{(1)} \end{bmatrix},$$
$$H_{N=8}^{(2)} = H_2 \otimes H_{N=4}^{(2)} = \begin{bmatrix} H_{N=4}^{(2)} & H_{N=4}^{(2)} \\ H_{N=4}^{(2)} & -H_{N=4}^{(2)} \end{bmatrix}. \quad (13)$$

Matrix $H_{N=8}^{(1)}$ is the WH matrix of size N=8, i.e., $H_{N=8}^{(1)} = H_{N=8}$, represented in Appendix 2. Matrix $H_{N=8}^{(2)}$ is also represented in Appendix 2.

Square binary matrices $H_{N=8}^{(1)}$ and $H_{N=8}^{(2)}$ with size N=8 are orthogonal because binary matrices $H_{N=4}^{(1)}$ and $H_{N=4}^{(2)}$ are orthogonal matrices (10), (11), and (12). Also, the binary matrices $H_{N=8}^{(1)}$ and $H_{N=8}^{(2)}$ consist of different binary sequences, and no other similar binary sequences exist inside these matrices.

2. In the **second step**, use the circular rotation procedure between matrices to rotate the second (right) part of the matrices between matrices $H_{N=8}^{(1)}$ and $H_{N=8}^{(2)}$ (Appendix 1). The procedure results in two more binary matrices:

$$H_{N=8}^{(1/2)} = \begin{bmatrix} H_{N=4}^{(1)} & H_{N=4}^{(2)} \\ H_{N=4}^{(1)} & -H_{N=4}^{(2)} \end{bmatrix}, \quad H_{N=8}^{(2/1)} = \begin{bmatrix} H_{N=4}^{(2)} & H_{N=4}^{(1)} \\ H_{N=4}^{(2)} & -H_{N=4}^{(1)} \end{bmatrix}. \qquad (14)$$

These two matrices are represented in Appendix 2. Observe that these matrices are orthogonal square matrices with size N = 8 because of the orthogonality of matrices $H_{N=4}^{(1)}$ and $H_{N=4}^{(2)}$ sequences and because the matrices sequences are binary (Appendix 1). Matrices $H_{N=8}^{(1/2)}$ and $H_{N=8}^{(2/1)}$ are independent binary orthogonal square matrices, i.e., all of the binary sequences inside each matrix are orthogonal, but the sequences in the different matrices are not orthogonal. The matrices (14) will be enumerated as the matrix number three and matrix number four, respectively:

$$H_{N=8}^{(3)} = H_{N=8}^{(1/2)}, \quad H_{N=8}^{(4)} = H_{N=8}^{(2/1)}. \qquad (15)$$

There is one more noteworthy feature of all four matrices $H_{N=8}^{(1)}$, $H_{N=8}^{(2)}$, $H_{N=8}^{(3)}$, and $H_{N=8}^{(4)}$ (13), (14), and (15). All of the binary sequences in these matrices are different in elements or by different orders of pluses and minuses (Appendix 2).

3. In the **third step,** construct new matrices from the four matrices $H_{N=8}^{(1)}$, $H_{N=8}^{(2)}$, $H_{N=8}^{(3)}$ and, $H_{N=8}^{(4)}$, using the synchronized rotation of matrices sequences (Appendix 1), i. e., create the matrices

$$H_{N=8}^{(1)(\uparrow R/R)} = \begin{bmatrix} H_{N=4}^{(1)} & H_{N=4}^{(1)(\uparrow R)} \\ H_{N=4}^{(1)} & -H_{N=4}^{(1)(\uparrow R)} \end{bmatrix},$$

$$H_{N=8}^{(2)(\uparrow R/R)} = \begin{bmatrix} H_{N=4}^{(2)} & H_{N=4}^{(2)(\uparrow R)} \\ H_{N=4}^{(2)} & -H_{N=4}^{(2)(\uparrow R)} \end{bmatrix},$$

$$H_{N=8}^{(3)(\uparrow R/R)} = \begin{bmatrix} H_{N=4}^{(1)} & H_{N=4}^{(2)(\uparrow R)} \\ H_{N=4}^{(1)} & -H_{N=4}^{(2)(\uparrow R)} \end{bmatrix}, \qquad (16)$$

and $$H_{N=8}^{(4)(\uparrow R/R)} = \begin{bmatrix} H_{N=4}^{(2)} & H_{N=4}^{(1)(\uparrow R)} \\ H_{N=4}^{(2)} & -H_{N=4}^{(1)(\uparrow R)} \end{bmatrix}.$$

The total number of rotations is defined by the number of $H_{N=4}$ rows and equals four (including R = 0). The R = 0, 1, 2, 3, where R = 0 corresponds to matrices without rotation. After each rotation, new matrices appear consisting of new binary sequences. All of the new matrices are orthogonal because matrices $H_{N=4}^{(1)}$ and $H_{N=4}^{(2)}$ are orthogonal and because all of the matrix sequences are binary sequences (Appendix 1). Samples of these matrices for N=8 are represented in Appendix 2.

The synchronized rotation of all four matrices $H_{N=8}^{(1)\,(\uparrow R/R)}$, $H_{N=8}^{(2)\,(\uparrow R/R)}$, $H_{N=8}^{(3)\,(\uparrow R/R)}$, and $H_{N=8}^{(4)\,(\uparrow R/R)}$, results in a total of 16 binary orthogonal square matrices (including R = 0) with size N = 8. Thus, we obtain $Q_{N=8}$ = 16 square orthogonal binary matrices with a sequence length equal to N=8. All of these matrices consist of unique binary sequences. Therefore, the total number of different binary sequences with length N=8 equals Q_N x N =16 x 8 = 128. This number corresponds to the maximum number of different binary sequences with length N = 8, namely, $m_{N=8} = 2^{N-1} = 2^7 = 128$.

Upon completing the procedure, we have a family of $Q_{N=8}$ = 16 binary orthogonal matrices for N = 8.

a) The first two matrices ($H_{N=8}^{(1)}$ and $H_{N=8}^{(2)}$) were constructed by performing the Kronecker multiplication of Hadamard matrices $H^{(1)}_{N=4}$ and $H^{(2)}_{N=4}$ by matrix H_2 (13).

b) Matrices three and four ($H_{N=8}^{(3)}$ and $H_{N=8}^{(4)}$) were constructed using the circular rotation procedure between matrices $H_{N=8}^{(1)}$ and $H_{N=8}^{(2)}$ (14).

c) Matrices $H_{N=8}^{(5)}$, $H_{N=8}^{(6)}$,…, $H_{N=8}^{(16)}$ were generated using the synchronized circular rotation procedure inside matrices $H_{N=8}^{(1)}$, $H_{N=8}^{(2)}{}_{N=8}$, $H_{N=8}^{(3)}{}_{N=8}$, and $H_{N=8}^{(4)}$ (15), (16).

Because we generated all 16 binary orthogonal matrices from two different Hadamard matrices for N = 4, i.e., matrices $H_{N=4}^{(1)}$ and $H_{N=4}^{(2)}$, we call these matrices an expanded family of Hadamard orthogonal binary matrices for N = 8. This family also includes the binary orthogonal matrix Walsh-Hadamard (WH) $H_{N=8}$ as matrix number one $H_{N=8}^{(1)}$.

The same procedure can be used to build all of the orthogonal binary matrices for N = 16 using the same three general steps used to construct the matrices for N = 8 (13), (14), (15), and (16).

1. In the **first step**, perform Kronecker multiplication to multiply all 16 orthogonal matrices $H_{N=8}^{(1)}$, $H_{N=8}^{(2)}$, …, $H_{N=8}^{(15)}$, and $H_{N=8}^{(16)}$ by matrix H_2 as performed for N = 8 (13).

$$H_{N=16}^{(1)} = H_2 \otimes H_{N=8}^{(1)} = \begin{bmatrix} H_{N=8}^{(1)} & H_{N=8}^{(1)} \\ H_{N=8}^{(1)} & -H_{N=8}^{(1)} \end{bmatrix},$$

$$H_{N=16}^{(2)} = H_2 \otimes H_{N=8}^{(2)} = \begin{bmatrix} H_{N=8}^{(2)} & H_{N=8}^{(2)} \\ H_{N=8}^{(2)} & -H_{N=8}^{(2)} \end{bmatrix}, \qquad (17)$$

$$\ldots\ldots ,$$

$$\ldots\ldots ,$$

and $$H_{N=16}^{(16)} = H_2 \otimes H_{N=8}^{(16)} = \begin{bmatrix} H_{N=8}^{(16)} & H_{N=8}^{(16)} \\ H_{N=8}^{(16)} & -H_{N=8}^{(16)} \end{bmatrix}.$$

After this Kronecker multiplication procedure, there are $2^4 = 16$ binary square matrices with sequences of length N = 16. These square binary matrices are orthogonal binary matrices because all sixteen matrices $H_{N=8}^{(1)}$, $H_{N=8}^{(2)}$, …, $H_{N=8}^{(16)}$ are orthogonal matrices.

2. In the **second step**, use the circular rotation procedure between all 16 orthogonal matrices (17) (Appendix 1). This procedure is similar to procedure (14) for N = 8.

After completing this procedure, there are $2^4 \times 2^4 = 2^8 = 256$ orthogonal binary square matrices with sequences of length N = 16.

3. In the **third step**, create new matrices from all the 256 orthogonal square matrices using the synchronized rotation procedure inside the matrices (Appendix 1). This procedure is similar to the rotation procedure (16) for N = 8. The total number of rotations is defined as the number of matrix $H_{N=8}$ rows and equals 8. Thus, R = 0, 1, 2, …, 7, where R = 0 corresponds to matrices without rotation. After this procedure, each of the above 256 orthogonal binary matrices will correspond to 8 matrices, including matrices without a rotation.

After applying this synchronized rotation procedure to all 256 matrices, we will obtain $2^8 \times 2^3 = 2^{11}$ orthogonal binary square matrices with sequences of length N = 16. Thus, for N = 16, we have $Q_{N=16} = 2^{11} = 2048$ orthogonal binary square matrices with sequences of length N = 16. All of these matrices consist of only different binary sequences with length N = 16. And the total number of different sequences is $2^{11} \times 2^4 = 2^{15}$, which corresponds to the maximum number of different binary sequences with length N = 16, i.e., $m_N = 2^{N-1} = 2^{15}$. One can use a similar procedure to create orthogonal square binary matrices for N = 32, 64, 128, etc.

Thus, for $N = 2^n$ (n = 3, 4, 5, …), it is possible to create additional non-WH orthogonal binary square matrices with different sequences of length N. The total number of additional matrices can be found by dividing the total number of binary sequences, $m_N = 2^{N-1}$, by the number of rows in the matrices, $N = 2^n$. The total number of binary orthogonal square matrices for $N = 2^n$ (n = 3, 4, 5, …) is $Q_N = 2^{(N-(n+1))}$.

The primary properties of these orthogonal binary square matrices with size $N=2^n$ (n = 3, 4, 5, …) are:

a) All of the sequences inside each of the matrices are orthogonal. But sequences which belong to different matrices are not orthogonal.

b) All of the matrices consist of different sequences with length $N = 2^n$. Each sequence from the total number of sequences, $m_N = 2^{N-1}$, belongs to only one matrix.

c) For any $N = 2^n$ (n = 3, 4, 5, …), WH matrices $H_N^{(1)}$ (i.e., matrices number one) have n+1 groups (G_1, G_2, ….) (Appendix 2) of sequences that correspond to n+1 groups of periodic signals with zero cross correlation or with zero mutual access interference (MAI).[8]

d) For any $N = 2^n$ (n = 3, 4, 5, …), matrices $H_N^{(2)}$ (i.e., matrices number two) have n - 1 groups (G_1, G_2, ….) (Appendix 2) of sequences that correspond to n - 1 groups of periodic signals with zero cross correlation or with zero mutual access interference (MAI).[8]

Because all of these $Q_N = 2^{(N-(n+1))}$ orthogonal binary square matrices for $N = 2^n$ (n = 3, 4, 5, …) are related with two Hadamard matrices for N = 4 (n = 2) (10), (11) and (12), we call these matrices an expanded family of Hadamard matrices for $N = 2^n$ (n = 3, 4, 5, ….). And WH matrices $H_N = H_N^{(1)}$ for $N = 2^n$ (n = 3, 4, 5, …) are only a subset of this expanded family of orthogonal binary matrices.

Shortened Families of Hadamard Matrices

The total number of the expanded families of Hadamard matrices increases quickly with an increasing value of N. For instance, for N = 8, 16, and 32, the total number of binary orthogonal matrices are $Q_N = 2^{(N - (n+1))} = 2^4, 2^{11}$, and 2^{26}, respectively; i.e., Q_N = 16, 2048, …. In practice, it is more convenient to use the term "shortened" family of Hadamard matrices for $N = 2^n$ (n = 3, 4, 5, …) because of the fast growth rate.

A shortened family of Hadamard matrices consists of only four binary orthogonal matrices for any $N = 2^n$ (n = 3, 4, 5, …). One can construct these matrices in the following way:

$$H_N^{(1)} = (H_2 \otimes)^{n-2} \otimes H_{N=4}^{(1)},$$
$$H_N^{(2)} = (H_2 \otimes)^{n-2} \otimes H_{N=4}^{(2)},$$
$$H_N^{(3)} = H_N^{(1/2)},$$
and
$$H_N^{(4)} = H_N^{(2/1)}, \quad (18)$$

Where $H_{N=4}^{(1)}$ and $H_{N=4}^{(2)}$ are a WH matrix and a Hadamard back-circulant matrix for N = 4, respectively, i.e., matrix number one and matrix number two (10), (11) and (12). Matrices $H_N^{(3)} = H_N^{(1/2)}$ and $H_N^{(4)} = H_N^{(2/1)}$ are circular rotated matrices when the second (right) parts of the matrices are rotated between matrices $H_N^{(1)}$ and $H_N^{(2)}$ (14) (Appendix 1).

The formation procedure for shortened family matrices for $N = 2^n = 8$ (n = 3) is

$$H_{N=8}^{(1)} = (H_2 \otimes)^{n-2} \otimes H_{N=4}^{(1)} = H_2 \otimes H_{N=4}^{(1)} = \begin{bmatrix} H_{N=4}^{(1)} & H_{N=4}^{(1)} \\ H_{N=4}^{(1)} & -H_{N=4}^{(1)} \end{bmatrix}$$

and (19)

$$H_{N=8}^{(2)} = (H_2 \otimes)^{n-2} \otimes H_{N=4}^{(2)} = H_2 \otimes H_{N=4}^{(2)} = \begin{bmatrix} H_{N=4}^{(2)} & H_{N=4}^{(2)} \\ H_{N=4}^{(2)} & -H_{N=4}^{(2)} \end{bmatrix},$$

Where $H_{N=4}^{(1)}$ is a WH matrix and $H_{N=4}^{(2)}$ is a Hadamard back-circulant matrix for N = 4 (10), (11), and (12).

After performing the circular rotation procedure (14) between matrices (19), we obtain

$$H_{N=8}^{(3)} = H_{N=8}^{(1/2)} = \begin{bmatrix} H_{N=4}^{(1)} & H_{N=4}^{(2)} \\ H_{N=4}^{(1)} & -H_{N=4}^{(2)} \end{bmatrix}$$

and (20)

$$H_{N=8}^{(4)} = H_{N=8}^{(2/1)} = \begin{bmatrix} H_{N=4}^{(2)} & H_{N=4}^{(1)} \\ H_{N=4}^{(2)} & -H_{N=4}^{(1)} \end{bmatrix}.$$

Samples of matrices $H_{N=8}^{(1)}$, $H_{N=8}^{(2)}$, $H_{N=8}^{(3)}$, and $H_{N=8}^{(4)}$ are presented in Appendix 2. Appendix 2 also contains samples of matrices $H_{N=8}^{(5)} = H_{N=8}^{(3)\,(\uparrow R/R=1)}$ and $H_{N=8}^{(6)} = H_{N=8}^{(3)\,(\uparrow R/R=2)}$.

For the $N = 2^n = 16$ ($n = 4$) case, we have

$$H_{N=16}^{(1)} = (H_2 \otimes)^{n-2} \otimes H_{N=4}^{(1)} = H_2 \otimes (H_2 \otimes H_{N=4}^{(1)}) = H_2 \otimes \begin{bmatrix} H_{N=4}^{(1)} & H_{N=4}^{(1)} \\ H_{N=4}^{(1)} & -H_{N=4}^{(1)} \end{bmatrix} =$$

$$= \begin{bmatrix} H_{N=8}^{(1)} & H_{N=8}^{(1)} \\ H_{N=8}^{(1)} & -H_{N=8}^{(1)} \end{bmatrix} = \begin{bmatrix} H_{N=4}^{(1)} & H_{N=4}^{(1)} & H_{N=4}^{(1)} & H_{N=4}^{(1)} \\ H_{N=4}^{(1)} & -H_{N=4}^{(1)} & H_{N=4}^{(1)} & -H_{N=4}^{(1)} \\ H_{N=4}^{(1)} & H_{N=4}^{(1)} & -H_{N=4}^{(1)} & -H_{N=4}^{(1)} \\ H_{N=4}^{(1)} & -H_{N=4}^{(1)} & -H_{N=4}^{(1)} & H_{N=4}^{(1)} \end{bmatrix},$$

and (21)

$$H_{N=16}^{(2)} = (H_2 \otimes)^{n-2} \otimes H_{N=4}^{(2)} = H_2 \otimes (H_2 \otimes H_{N=4}^{(2)}) = H_2 \otimes \begin{bmatrix} H_{N=4}^{(2)} & H_{N=4}^{(2)} \\ H_{N=4}^{(2)} & -H_{N=4}^{(2)} \end{bmatrix} =$$

$$= \begin{bmatrix} H_{N=8}^{(2)} & H_{N=8}^{(2)} \\ H_{N=8}^{(2)} & -H_{N=8}^{(2)} \end{bmatrix} = \begin{bmatrix} H_{N=4}^{(2)} & H_{N=4}^{(2)} & H_{N=4}^{(2)} & H_{N=4}^{(2)} \\ H_{N=4}^{(2)} & -H_{N=4}^{(2)} & H_{N=4}^{(2)} & -H_{N=4}^{(2)} \\ H_{N=4}^{(2)} & H_{N=4}^{(2)} & -H_{N=4}^{(2)} & -H_{N=4}^{(2)} \\ H_{N=4}^{(2)} & -H_{N=4}^{(2)} & -H_{N=4}^{(2)} & H_{N=4}^{(2)} \end{bmatrix},$$

Where $H_{N=4}^{(1)}$ is a WH matrix and $H_{N=4}^{(2)}$ is a Hadamard back-circulant matrix for N = 4 (10), (11), and (12).

After performing a circular rotation of the second (right) parts of the matrices (14), we have

$$H_{N=16}^{(3)} = H_{N=16}^{(1/2)} = \begin{bmatrix} H_{N=8}^{(1)} & H_{N=8}^{(2)} \\ H_{N=8}^{(1)} & -H_{N=8}^{(2)} \end{bmatrix} = \begin{bmatrix} H_{N=4}^{(1)} & H_{N=4}^{(1)} & H_{N=4}^{(2)} & H_{N=4}^{(2)} \\ H_{N=4}^{(1)} & -H_{N=4}^{(1)} & H_{N=4}^{(2)} & -H_{N=4}^{(2)} \\ H_{N=4}^{(1)} & H_{N=4}^{(1)} & -H_{N=4}^{(2)} & -H_{N=4}^{(2)} \\ H_{N=4}^{(1)} & -H_{N=4}^{(1)} & -H_{N=4}^{(2)} & H_{N=4}^{(2)} \end{bmatrix},$$

and (22)

$$H_{N=16}^{(4)} = H_{N=16}^{(2/1)} = \begin{bmatrix} H_{N=8}^{(2)} & H_{N=8}^{(1)} \\ H_{N=8}^{(2)} & -H_{N=8}^{(1)} \end{bmatrix} = \begin{bmatrix} H_{N=4}^{(2)} & H_{N=4}^{(2)} & H_{N=4}^{(1)} & H_{N=4}^{(1)} \\ H_{N=4}^{(2)} & -H_{N=4}^{(2)} & H_{N=4}^{(1)} & -H_{N=4}^{(1)} \\ H_{N=4}^{(2)} & H_{N=4}^{(2)} & -H_{N=4}^{(1)} & -H_{N=4}^{(1)} \\ H_{N=4}^{(2)} & -H_{N=4}^{(2)} & -H_{N=4}^{(1)} & H_{N=4}^{(1)} \end{bmatrix},$$

Where $H_{N=4}^{(1)}$ is a WH matrix and $H_{N=4}^{(2)}{}_{N=4}$ is a Hadamard back-circulant matrix for N = 4 (9), (10), and (11). Using the same approach shown above, one may construct shortened families of Hadamard matrices for N = 32, 64, …, etc.

Following from (19), (20), (21) and (22), when constructing a shortened family of Hadamard matrices for any $N = 2^n$ ($n = 3, 4, 5, \ldots$), only two Hadamard orthogonal binary matrices are used: matrices $H_{N=4}^{(1)}$ and $H_{N=4}^{(2)}$. And with a circular rotation of the second (right) parts of matrices $H_N^{(1)}$ and $H_N^{(2)}$, it is possible to obtain matrices $H_N^{(3)}$ and $H_N^{(4)}$.

CONCLUSION

We show that for any $N = 2^n$ ($n = 3, 4, 5, \ldots$) orthogonal binary matrices similar to orthogonal binary Walsh-Hadamard matrices exist with size N x N. The number of these matrices, Q_N, grows fast with an increasing value of N and equals $Q_N = 2^{(N-(n+1))}$.

All newly identified orthogonal binary matrices, including the WH matrices, for any $N = 2^n$ ($n = 3, 4, 5, \ldots$) can be called an expanded family of Hadamard matrices because they emerge from two fundamental Hadamard matrices for N = 4. Based on Kronecker multiplication followed by a circular rotation of the matrix binary sequences, the formation procedure of these orthogonal binary matrices for any $N = 2^n$ ($n =3, 4, 5, \ldots$) is presented.

The shortened family of Hadamard matrices for any $N = 2^n$ ($n =3, 4, 5, \ldots$) consists of only four orthogonal binary matrices, including the WH matrices. The construction procedure of the shortened family of Hadamard matrices for any $N = 2^n$ ($n =3, 4, 5, \ldots$) also is presented.

APPENDIX 1: PROCEDURE DEFINITIONS

Circular Rotation of Matrix Sequences

The definition of the matrix sequence rotation is associated with the circular rotation, or circular permutation, of sequences with length N, represented as the matrix M_N.

Consider four numeric sequences with a length of N:

$$A = \{a_i\} = \{a_0, a_1, \ldots, a_{N-1}\}, \qquad B = \{b_i\} = \{b_0, b_1, \ldots, b_{N-1}\},$$
$$C = \{c_i\} = \{c_0, c_1, \ldots, c_{N-1}\}, \quad \text{and} \quad D = \{d_i\} = \{d_0, d_1, \ldots, d_{N-1}\}, \tag{1A}$$

Where N is a positive integer ($N = 1, 2, 3, \ldots$).

Without rotation, matrix M_N, $M_N = M_N^{(\uparrow R=0)}$, may be represented as

$$M_N = M_N^{(\uparrow R=0)} = \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} \begin{matrix} k\downarrow \\ 0 \\ 1 \\ 2 \\ 3. \end{matrix} \tag{2A}$$

After one rotation step, we can represent matrix $M_N^{(\uparrow R=1)}$ as

$$M_N^{(\uparrow R=1)} = \begin{bmatrix} B \\ C \\ D \\ A \end{bmatrix} \begin{matrix} k\downarrow \\ 0 \\ 1 \\ 2 \\ 3, \end{matrix} \tag{3A}$$

Where N is the length of the matrix rows (length of the sequences), k is the matrix row ordering number (k = 0, 1, 2, 3), the symbol (↑R=0) represents zero circular (cyclic) rotation (i.e., the absence of sequence rotations), and the symbol (↑R = 1) represents a first step circular (cyclic) rotation. The number of matrix sequences m defines the total number of matrix rows k (k = 0, 1, 2. 3, 4, …, m-1), and in this example, m = 4.

From equation (3A), a circular rotation of one step (↑R = 1) is the upward shift of the matrix M_N sequences by one step. In this case, row k = 0 in equation (3A) corresponds to sequence B instead of sequence A as in equation (2A), row k = 1 corresponds to sequence C instead of sequence B, etc. Thus, during a rotation procedure, all of the sequences are shifted up by one step, and sequence A is shifted to the bottom row k = m-1 (in this case, m = 4), thus closing this circle. Due to its circular nature, we name the procedure a circular rotation.

The second rotation step (↑R = 2) corresponds to shifting the matrix M_N sequences up by two steps compared with the original matrix $M_N = M_N^{(\uparrow R=0)}$ (2A).

$$M_N^{(\uparrow R=2)} = \begin{bmatrix} C \\ D \\ A \\ B \end{bmatrix} \begin{matrix} k\downarrow \\ 0 \\ 1 \\ 2 \\ 3. \end{matrix} \tag{4A}$$

The cyclic rotation procedure of matrix M_N sequences continues using the same process until rotation step number m-1. After rotation step number m-1, for m = 4, matrix $M_N{}^{(\uparrow R=m-1)}{}_N$ is

And after step number m, matrix M_N equals matrix $M_N{}^{(\uparrow R=m)}{}_N$ and is identical to the original version without rotation: $M_N = M_N^{(\uparrow R=m)} = M_N^{(\uparrow R=0)}$ (2A).

$$M_N^{(\uparrow R=3)} = \begin{bmatrix} D \\ A \\ B \\ C \end{bmatrix} \begin{matrix} k\downarrow \\ 0 \\ 1 \\ 2 \\ 3. \end{matrix} \quad (5A)$$

The total number of rotation steps (including step number zero, i.e., when R = 0) is defined by the number of matrix M_N rows and equals m. I.e., the rotation step numbers are R = 0, 1, 2, 3, …, m-1. And in the R=0 case (no rotation), $M_N = M_N{}^{(\uparrow R=0)}{}_N = M_N{}^{(\uparrow R=m)}$ as described above.

If all the sequences from (1A) are orthogonal, i.e.,

$$\begin{matrix} A \times B = \sum a_i b_i = 0, & & A \times C = \sum a_i c_i = 0, \\ A \times D = \sum a_i d_i = 0, & & B \times C = \sum b_i c_i = 0, \\ B \times D = \sum b_i d_i = 0, & \text{and} & C \times D = \sum c_i d_i = 0, \end{matrix} \quad (6A)$$

Then the original matrix $M_N = M_N{}^{(\uparrow R=0)}{}_N$ (2A) is an orthogonal matrix. And in this case, one may observe that matrices $M_N{}^{(\uparrow R)}$ remain orthogonal matrices during the circular rotation procedure of any R step (R = 0, 1, 2, 3, …, m-1).

Synchronized Circular Rotation of Matrix Sequences

Consider matrix M_{2N}

$$M_{2N} = \begin{bmatrix} M_N & M_N \\ M_N & -M_N \end{bmatrix} = \begin{bmatrix} A & A \\ B & B \\ C & C \\ D & D \\ A & -A \\ B & -B \\ C & -C \\ D & -D \end{bmatrix}, \quad (7A)$$

Where $M_N = M_N{}^{(\uparrow R=0)}$ is the original matrix (2A) consisting of sequences A, B, C, and D of length N (1A).

A synchronized rotation of matrix $M_{2N}{}^{(\uparrow R/R)}$ is the simultaneous rotation of matrices M_N and $-M_N$ of the right half of matrix M_{2N} (7A),

$$M_{2N}{}^{(\uparrow R)} = \begin{bmatrix} M_N & M_N{}^{(\uparrow R)} \\ M_N & -M_N{}^{(\uparrow R)} \end{bmatrix}, \quad (8A)$$

Where $M_N^{(\uparrow R)}$ and $-M_N^{(\uparrow R)}$ are the synchronized circular rotated matrices M_N and $-M_N$ on R steps. I.e., the circular rotation of matrices M_N and $-M_N$ is performed simultaneously using the same number of steps (R) for both matrices. Thus, the synchronized circular rotation procedure is similar to the circular rotation procedure defined above (3A, 4A, and 5A); however, it is performed simultaneously for matrices M_N and $-M_N$, which belong to the right half of matrix M_{2N} (8A).

Following from (3A), the circular (cyclic) rotation of one step (R=1) corresponds to the upward circular (cyclic) shift of matrix M_N sequence by one step. The same procedure is performed simultaneously inside matrix $-M_N$. For instance, after the first synchronized rotation procedure step, we obtain

$$M_{2N}^{(\uparrow R/R=1)} = \begin{bmatrix} M_N & M_N^{(\uparrow R=1)} \\ M_N & -M_N^{(\uparrow R=1)} \end{bmatrix} = \begin{bmatrix} A & B \\ B & C \\ C & D \\ D & A \\ A & -B \\ B & -C \\ C & -D \\ D & -A \end{bmatrix}. \quad (9A)$$

After the second step of the synchronized rotation procedure, we obtain

$$M_{2N}^{(\uparrow R/R=2)} = \begin{bmatrix} M_N & M_N^{(\uparrow R=2)} \\ M_N & -M_N^{(\uparrow R=2)} \end{bmatrix} = \begin{bmatrix} A & C \\ B & D \\ C & A \\ D & B \\ A & -C \\ B & -D \\ C & -A \\ D & -B \end{bmatrix}. \quad (10A)$$

Note that the cyclic rotation of the sequences in matrices M_N and $-M_N$ is only performed within these matrices (9A, 10A). There is no rotation of any sequences between matrices M_N and $-M_N$.

The total number of synchronized circular rotation steps (R) is defined by numbers of matrix M_N rows and equals m (3A). I.e., the rotation step numbers are R = 0, 1, 2, 3, …, m-1. In the samples above (9A, 10A) m = 4. In the R = 0 (zero rotation) and R = m cases, $M_{2N} = M_{2N}^{(\uparrow R/R=0)} = M_{2N}^{(\uparrow R/R=m)}$ because this rotation procedure is circular (cyclic):

$$M_{2N} = M_{2N}^{(\uparrow R/R=0)} = M_{2N}^{(\uparrow R/R=m)} = \begin{bmatrix} M_N & M_N^{(\uparrow R=0)} \\ M_N & -M_N^{(\uparrow R=0)} \end{bmatrix} = \begin{bmatrix} A & A \\ B & B \\ C & C \\ D & D \\ A & -A \\ B & -B \\ C & -C \\ D & -D \end{bmatrix}. \quad (11A)$$

We observe that if sequences A, B, C, and D are orthogonal sequences (6A), then matrix M_{2N} for R = 0 (7A, 11A) is an orthogonal matrix. Matrix $M_{2N} = M_{2N}^{(\uparrow R/R=0)}$ is an orthogonal matrix only for R = 0, i.e., only in the absence of a synchronized circular (cyclic) rotation.

Notice that to obtain orthogonal matrices $M_{2N}^{(\uparrow R/R)}$ for any $R \neq 0$, sequences A, B, C, and D must satisfy the following conditions in addition to conditions (6A):

$$A^2 = B^2 = C^2 = D^2$$

$$A^2 = \sum a_i^2, \qquad B^2 = \sum b_i^2, \qquad (12A)$$
$$C^2 = \sum c_i^2, \quad \text{and} \quad D^2 = \sum d_i^2.$$

Observe that the binary sequences whose elements (a_i, b_i, c_i, d_i) have only two values, namely +/-1, satisfy the conditions in equation (12A). Binary sequences with length N always have $A^2 = B^2 = C^2 = D^2 = N$. This result means that if matrices M_N consist of binary orthogonal sequences, then matrices with a synchronized circular (cyclic) rotation $M_{2N}^{(\uparrow R/R)}$ remain orthogonal matrices for any R (R = 0, 1, 2, …, m-1).

Circular Rotation Procedure of Sequences Between Matrices

Consider the case when there are four numeric sequences with length N

$A = \{a_i\} = \{a_0, a_1, \ldots, a_{N-1}\}$, $B = \{b_i\} = \{b_0, b_1, \ldots, b_{N-1}\}$,

$C = \{c_i\} = \{c_0, c_1, \ldots, c_{N-1}\}$, and $D = \{d_i\} = \{d_0, d_1, \ldots, d_{N-1}\}$,

where $A \times B = \sum a_i b_i = 0$, and $C \times D = \sum c_{idi} = 0$, (13A)

but $A \times C = \sum a_i c_i \neq 0$, $A \times D = \sum a_i d_i \neq 0$,

$B \times C = \sum b_i c_i \neq 0$, and $B \times D = \sum b_i d_i \neq 0$.

Sequences A & B and C & D are orthogonal sequences, but sequences A & C, A & D, B & C, and B & D are not orthogonal sequences.

In this case, matrices M_{AB} and M_{CD} (11A),

$$M_{AB} = \begin{bmatrix} A & A \\ B & B \\ A & -A \\ B & -B \end{bmatrix}, \quad M_{CD} = \begin{bmatrix} C & C \\ D & D \\ C & -C \\ D & -D \end{bmatrix}, \qquad (14A)$$

are orthogonal matrices because sequences A & B and C & D are orthogonal sequences.

Matrices $M_{AB/CD}$ and $M_{CD/AB}$,

$$M_{AB/CD} = \begin{bmatrix} A & C \\ B & D \\ A & -C \\ B & -D \end{bmatrix}, \quad M_{CD/AB} = \begin{bmatrix} C & A \\ D & B \\ C & -A \\ D & -B \end{bmatrix}, \qquad (15A)$$

are called matrices with circularly rotated second (right) parts between matrices M_{AB} and M_{CD} (14A).

These matrices (15A) are orthogonal matrices when all their sequences (A, B, C, and D) are binary sequences. In this case, for all of the sequences, $A^2 = B^2 = C^2 = D^2 = N$ (12A), where N is the length of binary sequences A, B, C, and D. Equation (12A) and the orthogonality of sequences A & B and C & D (13A) are sufficient conditions to guarantee the orthogonality of matrices $M_{AB/CD}$ and $M_{CD/AB}$ despite the absence of orthogonality of sequences A & C, A & D, B & C, and B & D.

APPENDIX 2: SQUARE ORTHOGONAL MATRIX EXAMPLES

The following are examples of square orthogonal binary matrices for N = 8.

$$H_{N=8}^{(1)} = H_2 \otimes H_{N=4}^{(1)} = \begin{bmatrix} H_{N=4}^{(1)} & H_{N=4}^{(1)} \\ H_{N=4}^{(1)} & -H_{N=4}^{(1)} \end{bmatrix} = \begin{bmatrix} + & + & + & + & + & + & + & + \\ + & - & + & - & + & - & + & - \\ + & + & - & - & + & + & - & - \\ + & - & - & + & + & - & - & + \\ + & + & + & + & - & - & - & - \\ + & - & + & - & - & + & - & + \\ + & + & - & - & - & - & + & + \\ + & - & - & + & - & + & + & - \end{bmatrix} \begin{matrix} k\downarrow & G\downarrow \\ 0 & G_1 \\ 1 & G_2 \\ 2 & G_3 \\ 3 & \\ 4 & \\ 5 & G_4 \\ 6 & \\ 7 & \end{matrix} ,$$

$$H_{N=8}^{(2)} = H_2 \otimes H_{N=4}^{(2)} = \begin{bmatrix} H_{N=4}^{(2)} & H_{N=4}^{(2)} \\ H_{N=4}^{(2)} & -H_{N=4}^{(2)} \end{bmatrix} = \left[\begin{array}{cccc|cccc} + & + & + & - & + & + & + & - \\ + & - & + & + & + & - & + & + \\ + & + & - & + & + & + & - & + \\ + & - & - & - & + & - & - & - \\ + & + & + & - & - & - & - & + \\ + & - & + & + & - & + & - & - \\ + & + & - & + & - & - & + & - \\ + & - & - & - & - & + & + & + \end{array}\right] \begin{array}{cc} k\downarrow & G\downarrow \\ 0 & \\ 1 & G_1 \\ 2 & \\ 3 & \\ 4 & \\ 5 & G_2 \\ 6 & \\ 7 & \end{array},$$

$$H_{N=8}^{(3)} = H_{N=8}^{(1/2)} = \begin{bmatrix} H_{N=4}^{(1)} & H_{N=4}^{(2)} \\ H_{N=4}^{(1)} & -H_{N=4}^{(2)} \end{bmatrix} = \left[\begin{array}{cccc|cccc} + & + & + & + & + & + & + & - \\ + & - & + & - & + & - & + & + \\ + & + & - & - & + & + & - & + \\ + & - & - & + & + & - & - & - \\ + & + & + & + & - & - & - & + \\ + & - & + & - & - & + & - & - \\ + & + & - & - & - & - & + & - \\ + & - & - & + & - & + & + & + \end{array}\right] \begin{array}{c} k\downarrow \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array},$$

$$H_{N=8}^{(4)} = H_{N=8}^{(2/1)} = \begin{bmatrix} H_{N=4}^{(2)} & H_{N=4}^{(1)} \\ H_{N=4}^{(2)} & -H_{N=4}^{(1)} \end{bmatrix} = \left[\begin{array}{cccc|cccc} + & + & + & - & + & + & + & + \\ + & - & + & + & + & - & + & - \\ + & + & - & + & + & + & - & - \\ + & - & - & - & + & - & - & + \\ + & + & + & - & - & - & - & - \\ + & - & + & + & - & + & - & + \\ + & + & - & + & - & - & + & + \\ + & - & - & - & - & + & + & - \end{array}\right] \begin{array}{c} k\downarrow \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array},$$

$$H_{N=8}^{(5)} = H_{N=8}^{(3)\,(\uparrow R/R=1)} = \begin{bmatrix} H_{N=4}^{(1)} & H_{N=4}^{(2)\,(\uparrow R/R=1)} \\ H_{N=4}^{(1)} & -H_{N=4}^{(2)\,(\uparrow R/R=1)} \end{bmatrix} = \left[\begin{array}{cccc|cccc} + & + & + & + & + & - & + & + \\ + & - & + & - & + & + & + & - \\ + & + & - & - & + & - & - & - \\ + & - & - & + & + & + & - & + \\ + & + & + & + & - & + & - & - \\ + & - & + & - & - & - & - & + \\ + & + & - & - & - & + & + & + \\ + & - & - & + & - & - & + & - \end{array}\right] \begin{array}{c} k\downarrow \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array},$$

$$H_{N=8}^{(6)} = H_{N=8}^{(3)\,(\uparrow R/R=2)} = \begin{bmatrix} H_{N=4}^{(1)} & H_{N=4}^{(2)\,(\uparrow R/R=2)} \\ H_{N=4}^{(1)} & -H_{N=4}^{(2)\,(\uparrow R/R=2)} \end{bmatrix} = \begin{bmatrix} + & + & + & + & + & + & + & - \\ + & - & + & - & + & - & - & - \\ + & + & - & - & + & + & - & + \\ + & - & - & + & + & - & + & + \\ + & + & + & + & - & - & - & + \\ + & - & + & - & - & + & + & + \\ + & + & - & - & - & - & + & - \\ + & - & - & + & - & + & - & - \end{bmatrix} \begin{matrix} k\downarrow \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{matrix}$$

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