wjert, 2024, Vol. 10, Issue 8, 66-72. *Original Article* **ISSN 2454-695X**

World Journal of Engineering Research and Technology
WJERT
www.wjert.org
SJIF Impact Factor: 7.029

APPLICATIONS OF LINEAR CANONICAL-MELLIN TRANSFORM

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Article Received on 12/06/2024 **Article Revised on 02/07/2024** **Article Accepted on 23/07/2024**

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ABSTRACT

This article aims to introduce linear canonical-Mellin transform by defining proper testing function space. Then, we prove some of its differential properties which are very useful in solving differential equations. We also find the linear canonical-Mellin transform of two-dimensional Mexican hat wavelet to demonstrate the applicability of differentiation properties.

KEYWORDS: Linear canonical transform, Mellin transform, Testing function space, Mexican hat wavelet.

1. INTRODUCTION

Linear canonical transform (LCT) has been proven to be a very powerful tool in signal processing. Several properties, including differentiation properties, of LCT have been studied extensively in theory and applications both.[1]-[3] With the help of differentiation properties of LCT, one dimensional Mexican hat wavelet has been studied.[4]-[5]

The LCT is defined as.

$$L_A[f](u) = \Phi(u) = \begin{cases} \displaystyle\int_{-\infty}^{\infty} f(t)\, K_A(u,t)\, dt, & b \neq 0 \\ \sqrt{d}\, e^{j\frac{cd}{2}u^2} f(du), & b = 0 \end{cases}$$

where the LCT kernel $K_A(u,t)$ is given by the operator $K_A(u,t) = \frac{1}{\sqrt{j2\pi b}} e^{\frac{j}{2}\left[\frac{a}{b}t^2 - \left(\frac{2}{b}\right)tu + \frac{d}{b}u^2\right]}$ and parameters a, b, c, d are real numbers satisfying $ad - bc = 1$. On condition that the parameters satisfy $b = 0$, the LCT is essentially a scaling and chirp multiplication operations. Without loss of generality, we therefore focus mainly on the LCT in the case of $b \neq 0$. In that case, the inverse LCT is

$$f(t) = \sqrt{\frac{j}{2\pi b}} \int_{-\infty}^{\infty} \Phi(u)\, e^{-\frac{j}{2}\left(\frac{a}{b}t^2 - \frac{2}{b}tu + \frac{d}{b}u^2\right)} du$$

The Mellin transform is developed by Mellin (1854-1933) for the study of the gamma function, hypergeometric function, Dirichlet series, the Riemann zeta function and for the solution of partial differential equation.[6] It is defined as.

$M[f;s] \equiv F(s) = \int_0^\infty f(x)\, x^{s-1} dx$

The aim of this paper is to study the differentiation properties of LCMT and their application.

1. Linear Canonical-Mellin Transform (LCMT)

2.1 ***Definition***: The conventional Linear Canonical-Mellin transform is defined as follows:

$L_A M\{f(t,x)\} = F^A M(u,s) = \int_{-\infty}^{\infty} \int_0^\infty f(t,x)\, K(t,x,u,s)\, dt dx$

where $K(t,x,u,s) = \sqrt{\frac{1}{2j\pi b}}\, e^{\frac{j}{2}\left[\frac{a}{b}t^2 - \left(\frac{2}{b}\right)tu + \frac{d}{b}u^2\right]} x^{s-1}$, $b \neq 0, s > 0$.

Inverse of LCMT is given by

$(t,x) = \frac{1}{2\pi} \sqrt{\frac{j}{2\pi b}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F^A M(u,s)\, e^{-\frac{j}{2}\left(\frac{a}{b}t^2 - \frac{2}{b}tu + \frac{d}{b}u^2\right)} x^{-s} du ds$

2.2 The Testing Function Space $E(R^n)$

An infinitely differentiable complex valued smooth function φ on R^n belongs to $E(R^n)$, if for each compact set $K \subset S_a$, $I \subset S_b$,

where $S_a = \{t : t \in R^n, |t| \leq a, a > 0\}$ and $S_b = \{x : x \in R^n, |x| \leq b, b > 0\}$, $K, I \in R^n$,

$\gamma_{E,l,q} = \sup_{\substack{t \in K \\ x \in I}} \left| D_t^l D_x^q \varphi(t,x) \right| < \infty$, $\quad l, q = 0,1,2, - - -$

Thus $E(R^n)$ will denote the space of all $\varphi \in E(R^n)$ with support contained in S_a and S_b. Moreover, we say that f is a linear canonical-Mellin transformable if it is a member of E^*, the dual space of E.

2.3 Distributional Generalized Linear Canonical-Mellin Transform

The distributional Linear Canonical-Mellin transform of $f(t,x) \in E^*(R^n)$ is defined by

$L_A M\{f(t,x)\} = F^A M(u,s) = \langle f(t,x), K(t,x,u,s)\rangle$

(2.1.1)

where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $ad - bc = 1$ and

$K(t,x,u,s) = \sqrt{\frac{1}{2j\pi b}}\, e^{\frac{j}{2}\left[\frac{a}{b}t^2 - \left(\frac{2}{b}\right)tu + \frac{d}{b}u^2\right]} x^{s-1}$, $b \neq 0, s > 0.$

The right-hand side of (2.1.1) is meaningful because $K(t,x,u,s) \in E$ and $f(t,x) \in E^*$.

2. Differentiation Properties of LCMT

Property 1: $L_A M\left\{\frac{\partial}{\partial t} f(t,x)\right\} = L_A M\{f_t(t,x)\} = -j\left(cu + ja\frac{\partial}{\partial u}\right) F^A M(u,s)$

Proof: We have, by definition

$L_A M\{f_t(t,x)\}$

$= \sqrt{\frac{1}{2j\pi b}} \int_{-\infty}^{\infty}\int_0^{\infty} f_t(t,x)\, e^{\frac{j}{2}\left(\frac{a}{b}t^2 - \frac{2}{b}tu + \frac{d}{b}u^2\right)} x^{s-1} dt dx$

$= \sqrt{\frac{1}{2j\pi b}} \int_0^{\infty} x^{s-1}\left[\int_{-\infty}^{\infty} f_t(t,x) e^{\frac{j}{2}\left(\frac{a}{b}t^2 - \frac{2}{b}tu + \frac{d}{b}u^2\right)} dt\right] dx$

$= \sqrt{\frac{1}{2j\pi b}} \int_0^{\infty} x^{s-1}\left\{\left[f(t,x) e^{\frac{j}{2}\left(\frac{a}{b}t^2 - \frac{2}{b}tu + \frac{d}{b}u^2\right)}\right]_{-\infty}^{\infty}\right.$

$\left. - \int_{-\infty}^{\infty} \frac{j}{2}\left(2\frac{a}{b}t - 2\frac{u}{b}\right) f(t,x) e^{\frac{j}{2}\left(\frac{a}{b}t^2 - \frac{2}{b}tu + \frac{d}{b}u^2\right)} dt\right\} dx$

$= -j\frac{a}{b}\sqrt{\frac{1}{2j\pi b}} \int_{-\infty}^{\infty}\int_0^{\infty} t f(t,x)\, e^{\frac{j}{2}\left(\frac{a}{b}t^2 - \frac{2}{b}tu + \frac{d}{b}u^2\right)} x^{s-1} dt dx +$

$j\frac{u}{b}\sqrt{\frac{1}{2j\pi b}} \int_{-\infty}^{\infty}\int_0^{\infty} f(t,x)\, e^{\frac{j}{2}\left(\frac{a}{b}t^2 - \frac{2}{b}tu + \frac{d}{b}u^2\right)} x^{s-1} dt dx$

Provided $f(t,x) e^{\frac{j}{2}\left(\frac{a}{b}t^2 - \frac{2}{b}tu + \frac{d}{b}u^2\right)}$ vanishes as $t \to -\infty$ and $t \to \infty$

$= -j\frac{a}{b} L_A M\{t f(t,x)\} + j\frac{u}{b} L_A M\{f(t,x)\}$

$= -j\frac{a}{b}\left(du + jb\frac{\partial}{\partial u}\right) F^A M(u,s) + j\frac{u}{b} F^A M(u,s)$

$= -j\left(\frac{adu}{b} + ja\frac{\partial}{\partial u} - \frac{u}{b}\right) F^A M(u,s)$

$= -j\left[\left(\frac{ad-1}{b}\right)u + ja\frac{\partial}{\partial u}\right] F^A M(u,s)$

$= -j\left(cu + ja\frac{\partial}{\partial u}\right) F^A M(u,s).$

Property 2: $L_A M\left\{\frac{\partial^n}{\partial t^n} f(t,x)\right\} = (-1)^n j^n \left(cu + ja\frac{\partial}{\partial u}\right)^n F^A M(u,s)$

Proof: By Mathematical Induction, the proof is obvious and hence omitted.

Property 3: $L_A M\left\{\frac{\partial}{\partial x} f(t,x)\right\} = L_A M\{f_x(t,x)\} = -(s-1)F^A M(u,s-1)$

Proof: By definition,

$$L_A M\{f_x(t,x)\}$$

$$= \sqrt{\frac{1}{2j\pi b}} \int_{-\infty}^{\infty} \int_0^{\infty} f_x(t,x)\, e^{\frac{j}{2}\left(\frac{a}{b}t^2 - \frac{2}{b}tu + \frac{d}{b}u^2\right)} x^{s-1} dtdx$$

$$= \sqrt{\frac{1}{2j\pi b}} \int_{-\infty}^{\infty} e^{\frac{j}{2}\left(\frac{a}{b}t^2 - \frac{2}{b}tu + \frac{d}{b}u^2\right)} \left\{[x^{s-1} f(t,x)]_0^{\infty} - \int_0^{\infty} (s-1)x^{s-2} f(t,x)dx\right\}dt$$

$$= -(s-1)\sqrt{\frac{1}{2j\pi b}} \int_{-\infty}^{\infty} \int_0^{\infty} f(t,x)\, e^{\frac{j}{2}\left(\frac{a}{b}t^2 - \frac{2}{b}tu + \frac{d}{b}u^2\right)} x^{(s-1)-1} dtdx$$

Provided $[x^{s-1} f(t,x)]$ vanishes as $x \to 0$ and $x \to \infty$

$$= -(s-1)F^A M(u,s-1).$$

Property 4: $L_A M\left\{\frac{\partial^n}{\partial x^n} f(t,x)\right\} = (-1)^n (s-1)(s-2)\ldots(s-n)F^A M(u,s-n).$

$$= (-1)^n \frac{\Gamma s}{\Gamma(s-n)} F^A M(u,s-n).$$

Proof: By Mathematical Induction, the proof is obvious and hence omitted.

Property 5: $L_A M\left\{\frac{\partial^2}{\partial t \partial x} f(t,x)\right\} = L_A M\{f_{tx}(t,x)\}$

$$= j(-1)^2(s-1)\left(cu + ja\frac{\partial}{\partial u}\right)F^A M(u,s-1)$$

Proof: By definition

$$L_A M\{f_{tx}(t,x)\}$$

$$= \sqrt{\frac{1}{2j\pi b}} \int_{-\infty}^{\infty} \int_0^{\infty} f_{tx}(t,x)\, e^{\frac{j}{2}\left(\frac{a}{b}t^2 - \frac{2}{b}tu + \frac{d}{b}u^2\right)} x^{s-1} dtdx$$

$$= \sqrt{\frac{1}{2j\pi b}} \int_{-\infty}^{\infty} e^{\frac{j}{2}\left(\frac{a}{b}t^2 - \frac{2}{b}tu + \frac{d}{b}u^2\right)} \left\{[x^{s-1} f_t(t,x)]_0^{\infty} - \int_0^{\infty} (s-1) f_t(t,x) x^{s-2} dx\right\}dt$$

$$= -(s-1)\sqrt{\frac{1}{2j\pi b}} \int_{-\infty}^{\infty} \int_0^{\infty} f_t(t,x)\, e^{\frac{j}{2}\left(\frac{a}{b}t^2 - \frac{2}{b}tu + \frac{d}{b}u^2\right)} x^{s-2} dtdx$$

Provided $[x^{s-1} f_t(t,x)]$ vanishes as $x \to 0$ and $x \to \infty$

$$= -(s-1)\sqrt{\frac{1}{2j\pi b}} \int_0^{\infty} x^{s-2} \left\{\left[e^{\frac{j}{2}\left(\frac{a}{b}t^2 - \frac{2}{b}tu + \frac{d}{b}u^2\right)} f(t,x)\right]_{-\infty}^{\infty} -\right.$$

$$\left. \int_{-\infty}^{\infty} \frac{j}{2}\left(2\frac{a}{b}t - 2\frac{u}{b}\right) f(t,x) e^{\frac{j}{2}\left(\frac{a}{b}t^2 - \frac{2}{b}tu + \frac{d}{b}u^2\right)} dt\right\} dx$$

$$= (-1)^2(s-1)\left\{j\frac{a}{b}\sqrt{\frac{1}{2j\pi b}}\int_{-\infty}^{\infty}\int_0^{\infty} tf(t,x)\, e^{\frac{j}{2}\left(\frac{a}{b}t^2-\frac{2}{b}tu+\frac{d}{b}u^2\right)} x^{(s-1)-1}dtdx - \right.$$

$$\left. j\frac{u}{b}\sqrt{\frac{1}{2j\pi b}}\int_{-\infty}^{\infty}\int_0^{\infty} f(t,x)\, e^{\frac{j}{2}\left(\frac{a}{b}t^2-\frac{2}{b}tu+\frac{d}{b}u^2\right)} x^{(s-1)-1}dtdx\right\}$$

$$= (-1)^2(s-1)\left\{j\frac{a}{b}\left(du+jb\frac{\partial}{\partial u}\right)F^A M(u,s-1) - j\frac{u}{b}F^A M(u,s-1)\right\}$$

$$= j(-1)^2(s-1)\left\{\frac{(ad-1)}{b}u + ja\frac{\partial}{\partial u}\right\}F^A M(u,s-1)$$

$$= j(-1)^2(s-1)\left(cu + ja\frac{\partial}{\partial u}\right)F^A M(u,s-1)$$

Property 6

$$L_A M\left\{\frac{\partial^{n+m}}{\partial t^n \partial x^m}f(t,x)\right\} = j^n(-1)^{n+m}\frac{\Gamma s}{\Gamma(s-m)}\left(cu+ja\frac{\partial}{\partial u}\right)^n F^A M(u,s-m)$$

Proof: The proof is obvious and hence omitted.

3. Application

In this section, we define 2D-Mexican hat wavelet and we find its LCMT as an immediate application of differentiation properties.

4.1 2D- Mexican Hat Wavelet

2D-Mexican hat wavelet is defined by

$$\varphi(t,x) = [1-(t^2+x^2)]e^{\frac{-(t^2+x^2)}{2}}$$

$$= \frac{\partial^4}{\partial t^2 \partial x^2}e^{\frac{-(t^2+x^2)}{2}} - t^2x^2e^{\frac{-(t^2+x^2)}{2}} \qquad (4.1.1)$$

We have expressed 2D-Mexican hat wavelet in derivative form in order to find its LCMT.

Result 1: $L_A M\{e^{-(At^2+Bx^2)}\}(u,s) = \frac{1}{2}\sqrt{\frac{1}{a+j2Ab}}\, e^{j\frac{d}{2b}u^2} e^{\frac{u^2}{j2ab-4Ab^2}}\ B^{\frac{-s}{2}}\,\Gamma\left(\frac{s}{2}\right)$ (4.1.2)

Result 2: $L_A M\{t^n x^n f(t,x)\}(u,s) = j^n\left(-jdu+b\frac{\partial}{\partial u}\right)^n F^A M(u,s+n).$ (4.1.3)

4.2 Application

Example: (LCMT of 2D-Mexican hat wavelet)

$L_A M\{\varphi(t,x)\}(u,s)$

$$= \frac{1}{\sqrt{a+jb}}e^{j\frac{(ac+bd)-1}{2(a^2+b^2)}u^2}\,\Gamma\left(\frac{s}{2}-1\right)\left(\frac{1}{2}\right)^{\frac{-s}{2}}\left(\frac{s-2}{4}\right)\left\{\left[\frac{-a^2+b^2+j2ab}{(a^2+b^2)^2}u^2 + \frac{a^2-jab}{(a^2+b^2)}\right] - s\right\}$$

Proof. Using Eq. (4.1.1), we obtain

$$L_AM\{\varphi(t,x)\}(u,s) = L_AM\left\{\frac{\partial^4}{\partial t^2 \partial x^2} e^{\frac{-(t^2+x^2)}{2}}\right\}(u,s) - L_AM\left\{t^2x^2 e^{\frac{-(t^2+x^2)}{2}}\right\}(u,s)$$

$$= \frac{\Gamma s}{\Gamma(s-2)}\left(-jcu + a\frac{\partial}{\partial u}\right)^2 L_AM\left\{e^{\frac{-(t^2+x^2)}{2}}\right\}(u,s-2)$$

$$+\left(-jdu + b\frac{\partial}{\partial u}\right)^2 L_AM\left\{e^{\frac{-(t^2+x^2)}{2}}\right\}(u,s+2)$$

Putting $A = B = \frac{1}{2}$ in (4.1.2), we get

$$L_AM\{\varphi(t,x)\}(u,s)$$

$$= \frac{\Gamma s}{\Gamma(s-2)}\left(-jcu + a\frac{\partial}{\partial u}\right)^2 \frac{1}{2}\sqrt{\frac{1}{a+jb}}\, e^{j\frac{d}{2b}u^2} e^{\frac{u^2}{j2ab-2b^2}} \left(\frac{1}{2}\right)^{\frac{-(s-2)}{2}} \Gamma\left(\frac{s-2}{2}\right)$$

$$+\left(-jdu + b\frac{\partial}{\partial u}\right)^2 \frac{1}{2}\sqrt{\frac{1}{a+jb}}\, e^{j\frac{d}{2b}u^2} e^{\frac{u^2}{j2ab-2b^2}} \left(\frac{1}{2}\right)^{\frac{-(s+2)}{2}} \Gamma\left(\frac{s+2}{2}\right)$$

$$= \frac{\Gamma s}{\Gamma(s-2)}\, \Gamma\left(\frac{s-2}{2}\right)\left(\frac{1}{2}\right)^{\frac{4-s}{2}} \frac{1}{\sqrt{a+jb}}\, e^{j\frac{(ac+bd)-1}{2(a^2+b^2)}u^2} \left[\frac{a^2-b^2-j2ab}{(a^2+b^2)^2}u^2 - \frac{a^2-jab}{(a^2+b^2)}\right]$$

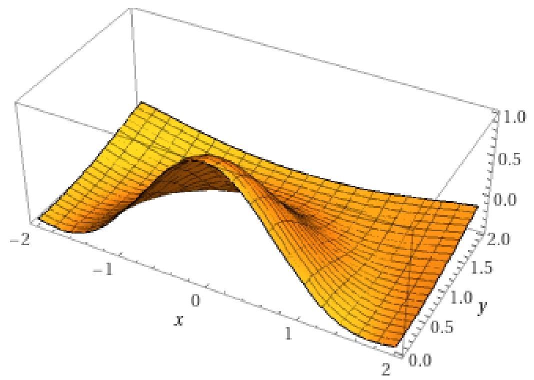
$$+\Gamma\left(\frac{s+2}{2}\right)\left(\frac{1}{2}\right)^{\frac{-s}{2}} \frac{1}{\sqrt{a+jb}}\, e^{j\frac{(ac+bd)-1}{2(a^2+b^2)}u^2} \left[\frac{-a^2+b^2+j2ab}{(a^2+b^2)^2}u^2 - \frac{b^2+jab}{(a^2+b^2)}\right]$$

$$= \frac{1}{\sqrt{a+jb}}\, e^{j\frac{(ac+bd)-1}{2(a^2+b^2)}u^2}\, \Gamma\left(\frac{s}{2}-1\right)\left(\frac{1}{2}\right)^{\frac{-s}{2}}\left(\frac{s-2}{4}\right) \times$$

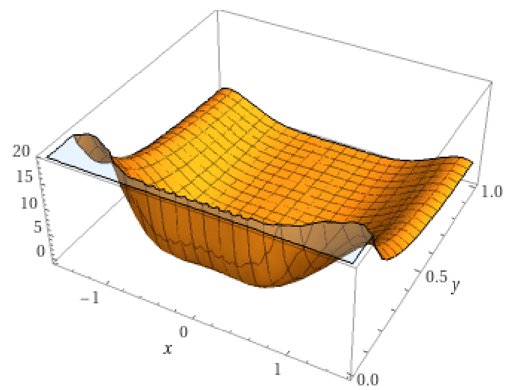
$$\left\{(s-1)\left[\frac{a^2-b^2-j2ab}{(a^2+b^2)^2}u^2 - \frac{a^2-jab}{(a^2+b^2)}\right] + s\left[\frac{-a^2+b^2+j2ab}{(a^2+b^2)^2}u^2 - \frac{b^2+jab}{(a^2+b^2)}\right]\right\}$$

$$= \frac{1}{\sqrt{a+jb}}\, e^{j\frac{(ac+bd)-1}{2(a^2+b^2)}u^2}\, \Gamma\left(\frac{s}{2}-1\right)\left(\frac{1}{2}\right)^{\frac{-s}{2}}\left(\frac{s-2}{4}\right)\left\{-s - \left[\frac{a^2-b^2-j2ab}{(a^2+b^2)^2}u^2 - \frac{a^2-jab}{(a^2+b^2)}\right]\right\}$$

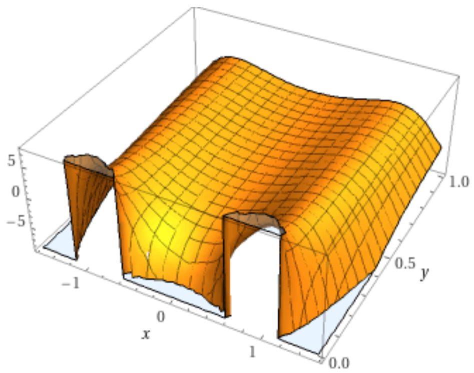
$$= \frac{1}{\sqrt{a+jb}}\, e^{j\frac{(ac+bd)-1}{2(a^2+b^2)}u^2}\, \Gamma\left(\frac{s}{2}-1\right)\left(\frac{1}{2}\right)^{\frac{-s}{2}}\left(\frac{s-2}{4}\right)\left\{\left[\frac{-a^2+b^2+j2ab}{(a^2+b^2)^2}u^2 + \frac{a^2-jab}{(a^2+b^2)}\right] - s\right\}$$



2D Mexican Hat Wavelet



Real part of LCMT of 2D Mexican Hat Wavelet for (a,b,c,d)= (-1/2, -1/2, 1,-1)



Imaginary part of LCMT of 2D Mexican Hat Wavelet (a,b,c,d)= (-1/2, -1/2, 1,-1)

4. CONCLUSION

We have proved various differentiation properties of LCMT and found LCMT of 2D-Mexican hat wavelet. These differentiation properties can further be used to solve generalized linear and non-linear differential equations.

REFERENCES

1. Wei, D., Ran, Q., Li, Y., Ma, J., & Tan, L. (2009). A convolution and product theorem for the linear canonical transform. *IEEE Signal Processing Letters*, *16*(10): 853-856.
2. Wei, D., Ran, Q., & Li, Y. (2012). A convolution and correlation theorem for the linear canonical transform and its application. *Circuits, Systems, and Signal Processing*, *31*: 301-312.
3. Goel, N., & Singh, K. (2013). A modified convolution and product theorem for the linear canonical transform derived by representation transformation in quantum mechanics. *International Journal of Applied Mathematics and Computer Science*, *23*(3): 685-695.
4. Zhang, Z. C. (2019). Linear canonical transform's differentiation properties and their application in solving generalized differential equations. *Optik*, *188*: 287-293.
5. Bahri, M., & Ashino, R. (2020). Solving generalized wave and heat equations using linear canonical transform and sampling formulae. In *Abstract and Applied Analysis* (Vol. 2020; 1: 1273194). Hindawi.
6. Eltayeb, H., & Kilicman, A. (2007). A note on Mellin transform and partial differential equations. *International Journal of Pure and Applied Mathematics*, *34*(4): 457.